Needle enzyme electrode based glucose diffusive transport measurement in a collagen gel and validation of a simulation model

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Mathematical model of mass transfer within a cylinder

The model envisages a cylindrical collagen gel with a radius of R without glucose immersed into a glucose solution with concentration c_0. Glucose diffuses into the gel and the concentration c follows Fick's Second Law and is a function of radial distance r and time t:

$$D\frac{\partial^2 c}{\partial r^2} + D\frac{\partial c}{r\partial r} = \frac{\partial c}{\partial t} \qquad \text{(S1)}$$

where D is the diffusion coefficient and the initial and boundary condition as: $c = 0$ for $t < 0$; $c\ (r = R) = c_0$ for $t \geq 0$.

The analytical solution for diffusion within a cylinder has the same mathematical expression as heat conduction within cylinder, known for long time.[19,27,28] For convenience, the dimensionless time $T = Dt/R^2$ is introduced. The solution to eqn. (S1) is given as:[19,27,28]

$$\frac{c}{c_0} = 1 - 2\sum_{n=1}^{\infty}\frac{J_0(\alpha_n r/R)}{\alpha_n J_1(\alpha_n)}\exp(-\alpha_n^2 T) \qquad \text{(S2)}$$

where n is an integer, and J_0 and J_1 are the zero order and first order of Bessel functions, respectively, with $J_0(\alpha_n) = 0$. As the needle electrode is located at the centre of the collagen cylinder, *i.e.*, $r = 0$, we have $J_0(0) = 1$ and eqn. (S2) is simplified further. Eqn. (S2) is a summation of an infinite series of exponential functions, which is not convenient for numerical simulation, therefore further analysis is necessary.

The terms of eqn. (S2) corresponding to n equal to 1, 2, 3 and 4, respectively, are shown in Fig. S1a. It can be seen that absolute values for the terms with n larger than 1 are very small for $T > 0.18$ and decreases with T and n increasing, respectively. This tendency is true for $n > 4$. Hence, for $T > 0.18$, the high order ($n > 1$) terms are negligible. Eqn. (S2) with n limited to 1, 2, 3 and 4 is represented in Fig. S1b for comparison. It can be seen that convergence of eqn. (S2) is rapid for large T and slow for small T. For $T > 0.18$, the four curves with different approximations are virtually identical therefore the first term well represents the whole solution. However, eqn. (S2) converges slowly for small T hence it is not practical for numerical simulations. An alternative solution for small T therefore is required.

For small T, the expressions for the solution to eqn. (S1) are different for r equals zero and nonzero.[19,27,28] Because the needle electrode is located at the centre of the cylinder, $r = 0$ is of interest. The first order approximation of the solution was derived using the Laplace transform method.[29] The first four terms of the solution are obtained with the Laplace transform method as:

$$\frac{c}{c_0}=\frac{1}{\sqrt{\pi T}}\exp(-\frac{1}{8T})\left[K_{\frac{1}{4}}(\frac{1}{8T})-\frac{T}{4}K_{\frac{3}{4}}(\frac{1}{8T})-\frac{7T^2}{32}K_{\frac{5}{4}}(\frac{1}{8T})-\frac{59T^3}{128}K_{\frac{7}{4}}(\frac{1}{8T})\right] \quad \text{(S3)}$$

where $K_{\frac{1}{4}}$, $K_{\frac{3}{4}}$, $K_{\frac{5}{4}}$ and $K_{\frac{7}{4}}$ are the fractional modified Bessel functions.

The first, second, third and fourth terms of eqn. (S3) are shown in Fig. S2a, respectively. It can be seen that absolute values of the second, third and fourth terms are very small for $T < 0.08$, and these values increase with T increasing. The first term, and accumulative terms up to the fourth for eqn. (S3) are shown in Fig. S2b for comparison. It can be seen that convergence of eqn. (S3) is rapid for small T but slow for large T. For $T < 0.08$, the four curves with different approximations are virtually identical therefore, the first term well represents the whole solution.

An effort was thus made to construct a solution function by combining the rapid convergent parts of eqns. (S2) and (S3). For a smooth connection, the two function curves are required to intersect and to have the same value or close values for first derivatives at the intersection point. However, the curves for eqn. (S2) with n limited to 1 and the first term of eqn. (S3) do not intersect. The minimum difference between these two curves is 0.021. Therefore, more terms are required in construction of an applicable solution function. Different combinations of eqn. (S2) with different n up to 4 and eqn. (S3) with different terms were tested. After a balance between complexity and accuracy, we constructed a normalised concentration function $c_n = c/c_0$ as following:

$$c_n = 1-2\sum_{n=1}^{3}\frac{\exp(-\alpha_n^2 T)}{\alpha_n J_1(\alpha_n)} \quad \text{for } T > 0.063, \quad \text{(S4a)}$$

$$c_n = \frac{1}{\sqrt{\pi T}}\exp(-\frac{1}{8T})\left(K_{\frac{1}{4}}(\frac{1}{8T})-\frac{T}{4}K_{\frac{3}{4}}(\frac{1}{8T})\right) \quad \text{for } T < 0.063. \quad \text{(S4b)}$$

When $T = 0.063$, eqns. (S4a) and (S4b) give the same values $c_n = 0.0358$. In fact when T varies from 0.059 to 0.067, the difference between eqns. (S4a) and (S4b) is less than 0.0001. In other words, eqns. (S4a) and (S4b) overlap over the range: $0.059 < T < 0.067$ with functional value difference less than 0.0001, which satisfies practical requirements.

The expressions of suitable solutions for the same differential equation are different for large and small T. The fast convergent parts of these two expressions overlap and form a practical function for an entire time range. From Fig. S1a it can be seen that the absolute values of terms, corresponding to n equal to 1, 2, 3 and 4 of eqn. (S2) decrease with T increasing and n increasing, respectively. At the joint point ($T = 0.063$), the absolute values of terms corresponding to n equal to 1, 2, 3 and 4, are 1.11, 0.156, 7.60×10^{-3} and 1.15×10^{-4}, respectively. This tendency is true for $n > 4$. Therefore, the terms with $n > 3$ are negligible for $T > 0.063$. From Fig. S2a it can be seen that the values of the first, second, third and fourth terms, the absolute value of the term of eqn. (S3) decreases with the term order number increasing and increases with T. At the joint point ($T = 0.063$) the absolute values of the first four terms values are 3.64×10^{-2}, 6.35×10^{-4}, 4.30×10^{-5} and 7.71×10^{-6}, respectively. The terms with term order number > 2 are negligible for $T < 0.063$. Therefore, it is reasonable that only the first three terms of eqn. (S2) and the first two terms of eqn. (S3) are used to construct the solution function, *i.e.*, eqn. (S4).

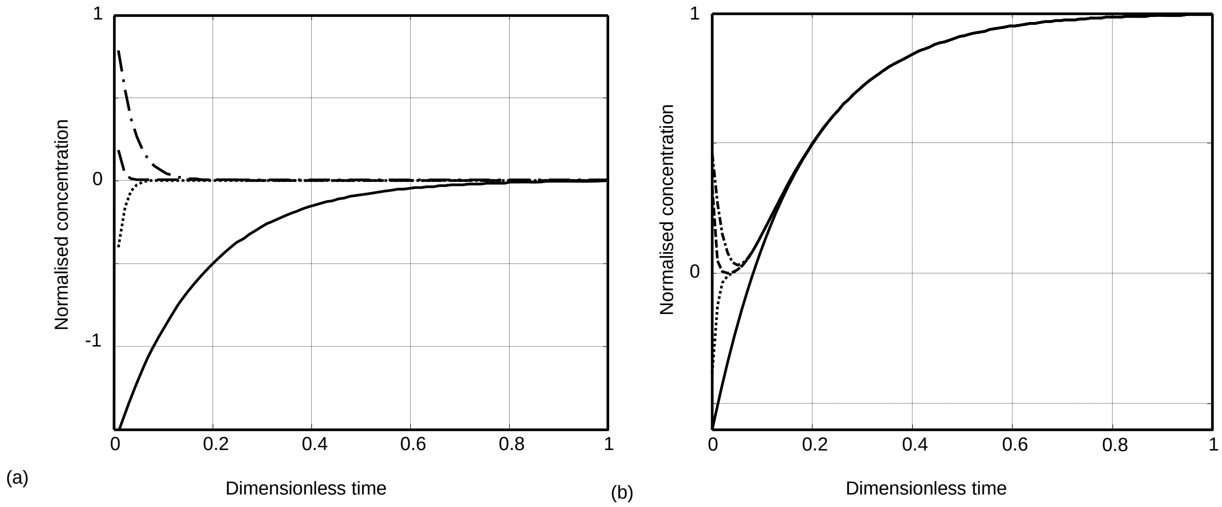

Fig. S1. (a) The terms corresponding to *n* equals to 1 (-), 2 (-·), 3 (··) and 4 (--) of eqn. (2), respectively. (b) Normalised concentration as eqn. (2) with *n* limited to 1 (-), 2 (-·), 3 (··) and 4 (--), respectively.

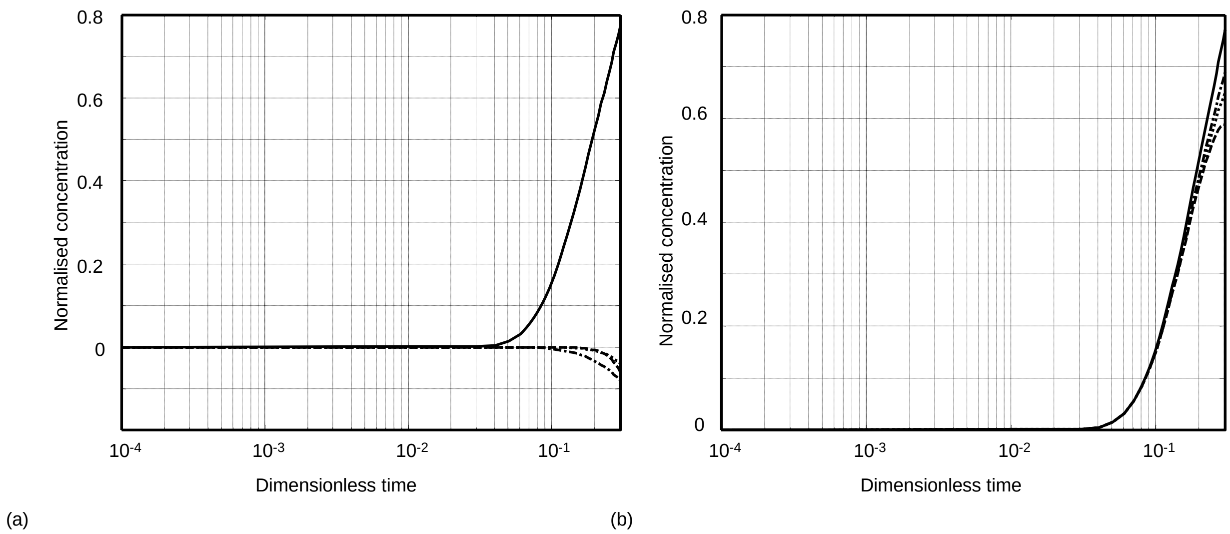

Fig. S2. (a) The first (-), second (-·), third (··) and fourth (--) terms of eqn. (3), respectively. (b) The first term (-), first two term (-·), the first three terms (··) and all four terms (--) of eqn. (3) are compared.