Electronic Supplementary Material (ESI) for RSC Advances.

Supporting Information for:

Harvesting vibrational energy with liquid-bridged electrodes: thermodynamics in mechanically and electrically driven RC-circuits

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TRADITIONAL RC-CIRCUIT

For comparison, we quickly recap some elementary results for traditional RC-circuits. The formal solution to eqn (4) for vanishing fixed charges $Q_{b/t}$ reads

$$q(t) = q_0 e^{-\int_0^t \frac{1}{RC(t')}dt'} + e^{-\int_0^t \frac{1}{RC(t')}dt'} \int_0^t dt' \frac{V(t')}{R} e^{\int_0^{t'} \frac{1}{RC(t'')}dt''}, \tag{S1}$$

which can for instance be found with variation of constants. A traditional RC-circuit has a constant capacitance $C(t) = C$, such that eqn (4) becomes a first order linear time-invariant theory,for which eqn (S1) simplifies to the well-known expression

$$q(t) = q_0 e^{-\frac{t}{RC}} + \int_0^t dt' \frac{V(t')}{R} e^{\frac{t'-t}{RC}}. \tag{S2}$$

In the case of a sinusoidal driving $V(t) = V_0 \sin \omega t$, we then find the potential over the load

$$V_{\rm L} = \frac{RCi\omega}{1 + RCi\omega} V_0, \tag{S3}$$

The phase angle between $V_{\rm L}$ and V_0 amounts to $\phi_{\rm L} = \tan^{-1}\left(\frac{1}{\omega RC}\right)$, from which we see that that there is a $\pi/2$ phase shift when increasing ω from $\omega \to 0$ ($\phi_{\rm L} = \pi/2$) to $\omega \to \infty$ ($\phi_{\rm L} = 0$). In Fig. S1 we show the power $P = \frac{|V_{\rm L}|^2}{R}$ dissipated over the resistor and relatedly, the energy $W = P \cdot T$ dissipated over the load during one cycle. The power drops to its half-value at the *cut-off* angular frequency $\omega RC = 1$. Relatedly, the work is seen to have a maximum $W(\omega = 1/RC) = \frac{1}{2}CV_0^2$ located at the same frequency.

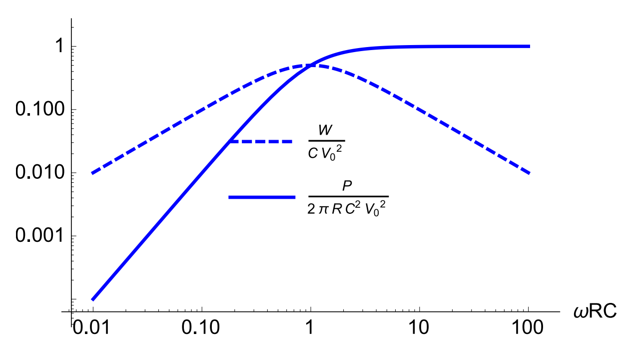

Figure S1. The work and power performed over a load resistor in a traditional RC-circuit.

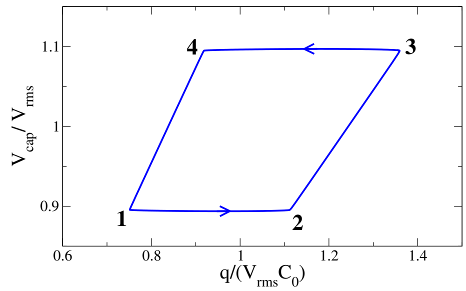

Figure S2. Definition of the state-points

Variable capacitance RC-circuit

The analytic solution Krupenkin and Taylor [1] present for the case of constant potential $V(t) = V_0$ with a sinusoidally varying capacitance $C_{t/b}(t) = C_0(1 + \cos\omega t)$, can be found by plugging in into eqn (S1). However, for the liquid bridge engine we are interested in, this expression for the capacitance is not correct since a vanishing capacitance would imply a broken bridge.

Dimensionless equations

The Kirchoff equation eqn (4) is brought in dimensionless form by writing $q(t) = \hat{q}(\hat{t})V_0C_0$, $t = \hat{t}\omega^{-1}$, $C_{b/t}(t) = C_{b/t}(0)\hat{C}_{b/t}(t)$ and $C_{tot} = C_0\hat{C}_{tot}$, and $V(t) = V_0\hat{V}(t)$. The choice of $C_0 \equiv C_{tot}(0)$ fixes $\hat{C}_{tot}^{-1} = \hat{C}_b^{-1}\left(1 + \frac{C_b(0)}{C_t(0)}\right)^{-1} + \hat{C}_t^{-1}\left(1 + \frac{C_t(0)}{C_b(0)}\right)^{-1}$. Defining $\alpha = \omega RC_0$, the Kirchoff equation reads

$$\alpha\frac{d\hat{q}(t)}{d\hat{t}} + \frac{\hat{q}(\hat{t})}{\hat{C}_{tot}(\hat{t})} = \hat{V}(t) \tag{S4}$$

$$\hat{q}(0) = 1 \tag{S5}$$

The energy harvested over the load, written in terms of the dimensionless variables, reads

$$W_{\mathrm{L}} = \int_0^{\hat{t}(T)} d\hat{t}\frac{d\hat{q}(\hat{t})}{d\hat{t}}(\hat{V}(t) - \frac{\hat{q}(\hat{t})}{\hat{C}_{tot}})V_0^2C_0 \tag{S6}$$

In a set-up **without external power source** the natural voltage scale is set by $\frac{Q_b}{C_b(0)}$, this equation is brought in dimensionless form by writing $q(t) = \hat{q}(\hat{t})Q_b$, $t = \hat{t}\omega^{-1}$, $C_{tot} = C_0\hat{C}_{tot}$, and $C_{b/t}(t) = C_{b/t}(0)\hat{C}_{b/t}(t)$ which gives

$$\alpha\frac{d\hat{q}(\hat{t})}{d\hat{t}} + \frac{\hat{q}(\hat{t})}{\hat{C}_{tot}(\hat{t})} = \frac{C_0}{C_b(0)}\left(\frac{1}{\hat{C}_b(\hat{t})} - \frac{1}{\hat{C}_t(\hat{t})}\right). \tag{S7}$$

Static limit of square wave driving

The square wave driving gives a trapezoidal enclosed area such that the harvested work amounts to

$$W_{\omega\to 0} = ((Q_2 - Q_1) + (Q_3 - Q_4))\,\Delta V \tag{S8}$$

The charges can be expressed as $Q_1 = V_lC_l$, $Q_2 = V_lC_h$, $Q_3 = V_hC_h$, and $Q_4 = V_hC_h$ with l low and h high. Combining gives

$$W_{\omega\to 0} = (V_l\,(C_h - C_l) + V_h\,(C_h - C_l))\,\Delta V \tag{S9}$$

Writing $\Delta C \equiv C_h - C_l$ and because $V_l = V_0 - \Delta V$ and $V_h = V_0 + \Delta V$ we find

$$W_{\omega\to 0} = 2\Delta C V_0 \Delta V \tag{S10}$$

We normalize this to $C_0 V_{\rm rms}^2$, using that the root mean square potential

$$\frac{W_{\omega\to 0}}{C_0 V_{\rm rms}^2} = 2\frac{\Delta C}{C_0}\frac{\frac{\Delta V}{V_0}}{1+\left(\frac{\Delta V}{V_0}\right)^2}, \tag{S11}$$

which is maximal at $\frac{\Delta V}{V_0} = 1$. For $L \ll l_c$ we can use that $A \sim \frac{1}{L}$ together with eqn (3) to find

$$\frac{\Delta C}{C_0} = 2\frac{\frac{\Delta L}{L_0}}{1-\left(\frac{\Delta L}{L_0}\right)^2}, \tag{S12}$$

which gives the final result

$$\frac{W_{\omega\to 0}}{C_0 V_{\rm rms}^2} = 4\frac{\frac{\Delta L}{L_0}}{1-\left(\frac{\Delta L}{L_0}\right)^2}\frac{\frac{\Delta V}{V_0}}{1+\left(\frac{\Delta V}{V_0}\right)^2}. \tag{S13}$$

[1] T. Krupenkin and J. A. Taylor, Nature Communications, 2011, **2**, 448.