Electronic Supplementary Material (ESI) for Soft Matter.

Supporting Information: Tuning the Mechanical Impedance of Disordered Networks for Impact Mitigation

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Velocity dependence of Young's modulus in DNMMs

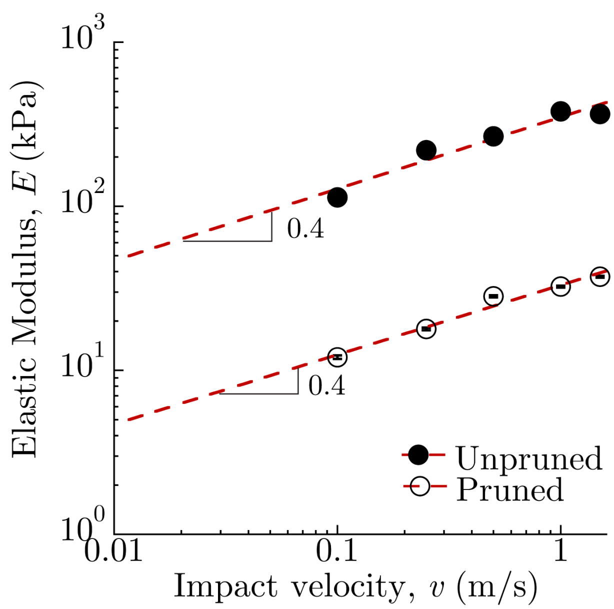

Figure 1: **Mechanical response to impact** of representative Shore-A 30 samples as a function of impact velocity, v, using a mechanical impactor. The slope of E as a function of v for both unpruned and pruned samples is the same. This indicates that the velocity dependence of the Young's modulus is intrinsic to the constituent material and not dependent on the network structure of the sample.

Compression of DNMMs along Principal Directions

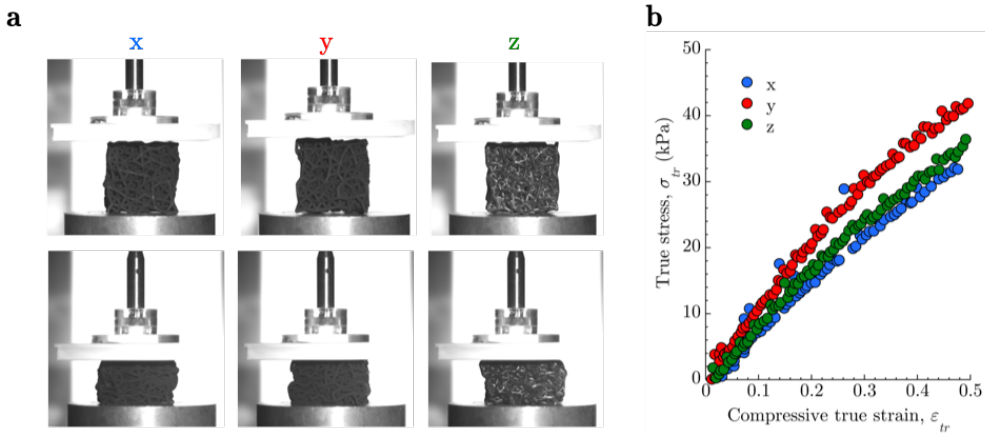

Figure 2: **Compression along principal axes of unpruned DNMMs.** a) Photographs of unpruned sample before and during uniaxial compression along the three principal directions. b) The corresponding stress vs. strain curves.

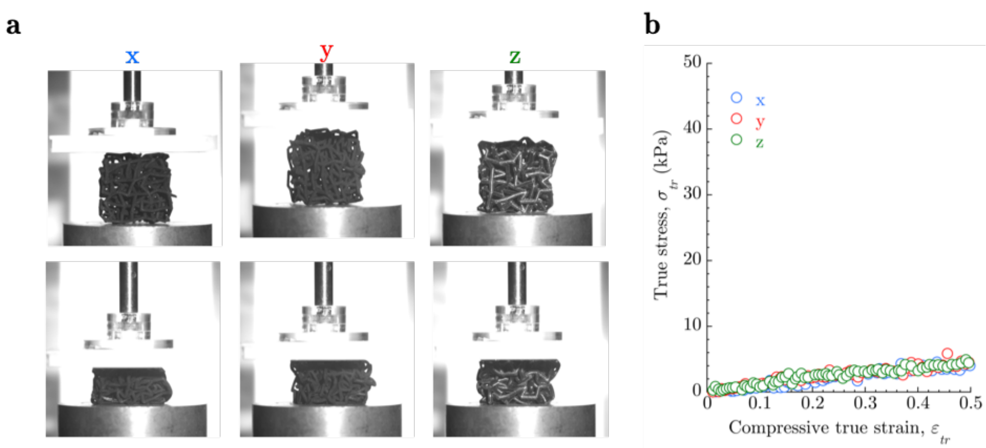

Figure 3: **Compression along principal axes of pruned DNMMs.** a) Photographs of unpruned sample before and during uniaxial compression along the three principal directions. b) The corresponding stress vs. strain curves.

Transverse strain measurement

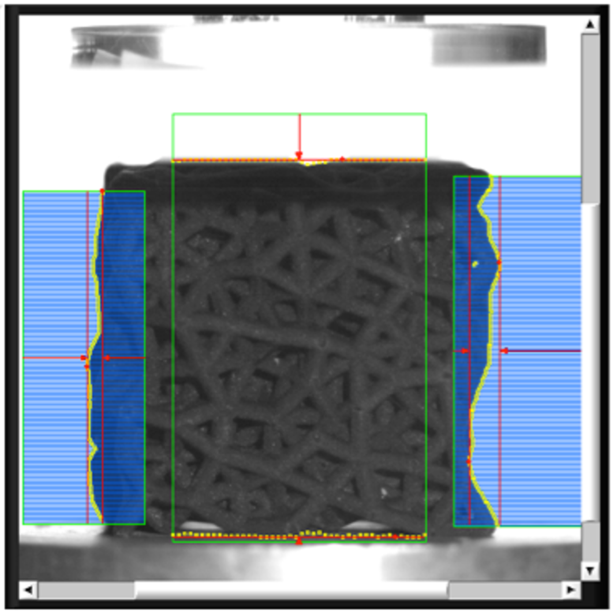

Figure 4: Edge detection for Poisson's ratio measurements.

A custom-designed LabVIEW edge-detection software was used to measure the changes in height average width of DNMM samples during the linear impact experiments (Fig. 4. Given the irregular edges of DNMM samples, the average width is estimated by the software at each frame and the transverse strain, $\varepsilon_{transverse}$, is calculated from the changes in average width.

Bulk vs. Shear modulus

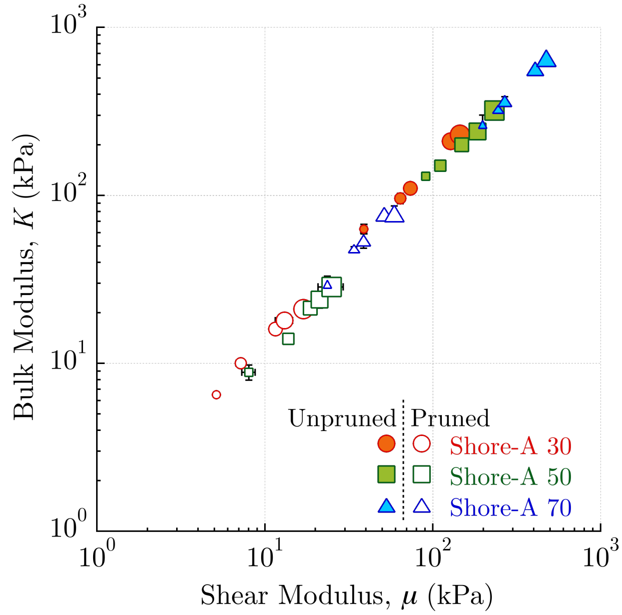

Figure 5: Bulk vs. Shear modulus. Tested for same unpruned structures, using three different printing materials at impact velocities between 0.1 − 1.5 m/s (increasing marker size indicates increasing impact velocity). Bulk and shear remain coupled for all tested samples in all tested conditions.

Transmitted Force Model

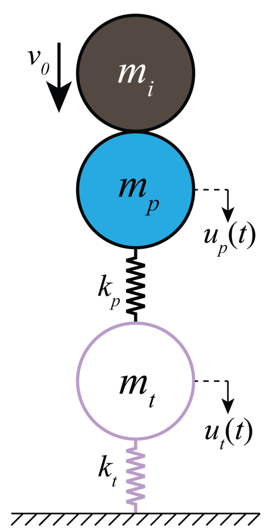

Figure 6: Impactor of mass m_i and impact velocity v_0 at the moment of contact with delicate target coupled to protective material

The time-dependent force applied to the DNMM measured by the target (load plate) can be derived based on the following expressions, derived in a similar manner by Argatov and Jokinen.[1] The acceleration ($\ddot{u}_p(t)$) of the protective material mass coupled to the mass of the impactor ($m = m_i + m_p$) is related to the spring constant of the protective material (k_p),

$$m\ddot{u}_p(t) = -k_p\big(u_p(t) - u_t(t)\big) \tag{1}$$

where $u_p(t)$ is the position of the top surface of the protective material and $u_t(t)$ is the position of the target. The stress imposed on the target is then,

$$-k_p\big(u_p(t) - u_t(t)\big) = A_t\sigma_t(0, t), \tag{2}$$

where A_t is the contact area of the target. Using D'Alembert solution, Eq.(1) and Eq.(2) become,

$$m\ddot{u}_p(t) = -k_p\big(u_p(t) - u_t(t)\big) \tag{3}$$

$$k_pu_p(t) - \frac{A_tE_t}{c_t}\dot{u}_t(t) - k_pu_t(t) = 0 \tag{4}$$

Taking the Laplace Transform of Eq.(3) and Eq.(4), we obtain,

$$u_p^*(s)\big(s^2 + \omega_0^2\big) - u_t^*(s)\omega_0^2 = v_0 \tag{5}$$

$$u_p^*(s)\omega_0^2 - u_t^*(s)\big(2\eta s + \omega_0^2\big) = 0 \tag{6}$$

where $\omega_0^2 = \frac{k_p}{m_p}$ and $2\eta = \frac{E_tA_t}{c_tm_p}$. Solving for $u_p^*(s)$ and $u_t^*(s)$ and taking the inverse Laplace Transform,

$$u_p(t) = \frac{v_0}{2\eta}\left(1 - e^{-\alpha t}cos(\omega t) + \lambda e^{-\alpha t}sin(\omega t)\right) \tag{7}$$

$$u_t(t) = \frac{v_0}{2\eta}\left(1 - e^{-\alpha t}cos(\omega t) - \beta e^{-\alpha t}sin(\omega t)\right) \tag{8}$$

where $\omega = \omega_0\sqrt{1-\gamma^2}$, $\alpha = \omega_0\gamma$, $\lambda = \frac{2-\gamma}{4\gamma\sqrt{1-\gamma^2}}$, $\beta = \frac{\gamma}{\sqrt{1-\gamma^2}}$, and $\gamma = \frac{\omega_0}{4\eta} = \frac{c_t}{2E_tA_t}\sqrt{m_fk_f}$

Since we are interested in solving for $F_t(t)$ and recalling that $F_t(t) = \frac{E_tA_t}{c_t}\dot{u}_t(t)$. Solving for $\dot{u}_t(t)$ using Eq.(8), we obtain,

$$\begin{aligned}\dot{u}_t(t) &= \frac{v_o}{2\eta}e^{-\alpha t}\Big((\alpha\beta+\omega)sin(\omega t) + (\alpha-\beta\omega)cos(\omega t)\Big)\\ &= \frac{v_o}{2\eta}e^{-\alpha t}\Big(\frac{\omega_0}{\sqrt{1-\gamma^2}}sin(\omega t)\Big)\end{aligned} \tag{9}$$

Submitting the appropriate constants into Eq.(9),

$$\begin{aligned}\dot{u}_t(t) &= \frac{v_o}{2\eta}\frac{\omega_0}{\sqrt{1-\gamma^2}}e^{-\alpha t}sin(\omega t)\\ &= 2v_o\frac{\gamma}{\sqrt{1-\gamma^2}}e^{-\alpha t}sin(\omega t)\end{aligned} \tag{10}$$

Substituting Eq.(10) into the force expression,

$$\begin{aligned}F_t(t) &= 2v_o\frac{E_tA_t}{c_t}\frac{\gamma}{\sqrt{1-\gamma^2}}e^{-\alpha t}sin(\omega t)\\ &= 2v_o\frac{E_tA_t}{c_t}\frac{\gamma}{\sqrt{1-\gamma^2}}e^{-\omega_0\gamma t}sin\Big(\omega_0 t\sqrt{1-\gamma^2}\Big)\end{aligned} \tag{11}$$

Since the sonic wave speed in the target is $c_t = \sqrt{\frac{E_t}{\rho_t}}$ and the acoustic impedance is $z_t = \sqrt{E_t\rho_t}$, Eq.(11) can be rewritten as,

$$F_t(t) = 2v_oz_tA_t\frac{\gamma}{\sqrt{1-\gamma^2}}e^{-\omega_0\gamma t}sin\Big(\omega_0 t\sqrt{1-\gamma^2}\Big) \tag{12}$$

Therefore, the stress experienced by the target is,

$$\sigma_t(t) = 2v_oz_t\frac{\gamma}{\sqrt{1-\gamma^2}}e^{-\omega_0\gamma t}sin\Big(\omega_0 t\sqrt{1-\gamma^2}\Big), \tag{13}$$

Peak transmitted force

Taking the derivative of Eq.(11) and setting it $= 0$,

$$F_t'(t) = 2v_o \frac{E_t A_t}{c_t} \frac{\omega_0 \gamma}{\sqrt{1-\gamma^2}} \left(\sqrt{1-\gamma^2} cos\left(\omega_0 t \sqrt{1-\gamma^2}\right) - \gamma sin\left(\omega_0 t \sqrt{1-\gamma^2}\right) \right) e^{-\omega_0 \gamma t} = 0 \tag{14}$$

This implies that either $e^{-\omega_0 \gamma t} = 0$ or $\sqrt{1-\gamma^2} cos\left(\omega_0 t \sqrt{1-\gamma^2}\right) - \gamma sin\left(\omega_0 t \sqrt{1-\gamma^2}\right) = 0$

The critical time at peak force (t_c) is,

$$\begin{aligned} \sqrt{1-\gamma^2} cos\left(\omega_0 t \sqrt{1-\gamma^2}\right) - \gamma sin\left(\omega_0 t \sqrt{1-\gamma^2}\right) = 0 \\ \implies tan\left(\omega_0 t \sqrt{1-\gamma^2}\right) = \frac{\sqrt{1-\gamma^2}}{\gamma} \\ t_c = \frac{1}{\omega_0 \sqrt{1-\gamma^2}} tan^{-1}\left(\frac{\sqrt{1-\gamma^2}}{\gamma}\right) \end{aligned} \tag{15}$$

Eq.(15) indicates that t_c increases with decreasing γ, in other words, when the acoustic impedance of the DNMM decreases (see below).

The maximum transmitted force, $F_{t,max} = F_t(t_c)$, can than be written as

$$F_{t,max} = v_o m \gamma \omega_0 \sqrt{\frac{1}{\gamma^2}} e^{-\frac{\gamma tan^{-1}\left(\frac{\sqrt{1-\gamma^2}}{\gamma}\right)}{\sqrt{1-\gamma^2}}} \tag{16}$$

Transmitted Impulse

To obtain the transmitted impulse, $F_t(t)$ needs to be integrated from $t = 0$ to $t = t_d$ where, t_d is the impact duration. t_d can be found by solving for the time using in $\dot{u}_t(t_d) = 0$. The duration of the impact is then,

$$t_d = \frac{\pi}{\omega} = \frac{\pi}{\omega_0 \sqrt{1-\gamma^2}} = \frac{\pi}{\sqrt{\frac{k_p}{m}} \sqrt{1-\phi^2 k_p m}} \tag{17}$$

Integrating $F_t(t)$ from $t = 0$ to $t = t_d$ we obtain the transmitted impulse,

$$I_t = \frac{v_o}{m}\left(1 + e^{-\frac{\pi\phi\sqrt{k_p m}}{\sqrt{1-\phi^2 k_p m}}}\right) \tag{18}$$

Which can also be recast as a function of E_p and ρ,

$$I_t = \frac{v_o}{\rho A_p l_p}\left(1 + e^{-\frac{A_p\pi\phi\sqrt{E_p\rho}}{\sqrt{1-A_p^2\phi^2 E_p\rho}}}\right) \tag{19}$$

Recalling that $\gamma = \frac{c_t}{2E_tA_t}\sqrt{mk_p}$ Eq.(11) can be simplified further,

$$\begin{aligned}
F_t(t) &= 2v_o\frac{E_tA_t}{c_t}\frac{\gamma}{\sqrt{1-\gamma^2}}e^{-\omega_0\gamma t}sin\left(\omega_0 t\sqrt{1-\gamma^2}\right)\\
&= 2v_o\frac{E_tA_t}{c_t}\frac{c_t}{2E_tA_t}\frac{\sqrt{mk_p}}{\sqrt{1-\gamma^2}}e^{-\omega_0\gamma t}sin\left(\omega_0 t\sqrt{1-\gamma^2}\right)\\
&= \frac{v_o\sqrt{\omega_0^2 m^2}}{\sqrt{1-\gamma^2}}e^{-\omega_0\gamma t}sin\left(\omega_0 t\sqrt{1-\gamma^2}\right)\\
F_t(t) &= \frac{v_o\omega_0 m}{\sqrt{1-\gamma^2}}e^{-\omega_0\gamma t}sin\left(\omega_0 t\sqrt{1-\gamma^2}\right)
\end{aligned} \tag{20}$$

Eq.(20) can be expressed in terms of k_p and m,

$$F_t(t) = \frac{v_o\sqrt{\frac{k_p}{m^3}}}{\sqrt{1-\phi^2 k_p m}}e^{-k_p\phi t}sin\left(t\sqrt{\frac{k_p}{m}}\sqrt{1-\phi^2 k_p m}\right) \tag{21}$$

where $\phi = \frac{c_t}{2E_tA_t}$.

Alternatively, Eq. (21) can be expressed in terms of ρ and E_p, recalling that $m = \rho A_p l_p$ and $k_p = \frac{E_pA_p}{l_p}$, where A_p and l_p are the crossectional area and length of the protective material, respectively.

$$F_t(t) = \frac{\frac{v_o}{A_p l_p^2}\sqrt{\frac{E_p}{\rho^3}}}{\sqrt{1-\phi^2 E_p A_p^2\rho}}e^{-\frac{E_fA_p}{l_p}\phi t}sin\left(\frac{t}{l_p}\sqrt{\frac{E_p}{\rho}}\sqrt{1-\phi^2 E_p A_p^2\rho}\right) \tag{22}$$

The γ parameter

γ is a dimensionless parameter defined as,

$$\gamma = \frac{\omega_0}{4\eta} = \frac{c_t\sqrt{k_p m_p}}{2E_t A_t} \tag{23}$$

The stiffness (k_p) and density (ρ_p) of the DNMM can be defined as,

$$\begin{aligned} k_p &= \frac{E_p A_p}{h_p} \\ m_p &= \rho_p A_p h_p \end{aligned} \tag{24}$$

where h_p is the height of the DNMM. Using Eq.(24) in Eq.(23),

$$\begin{aligned} \gamma &= \frac{c_t}{2E_t A_t}\sqrt{\frac{E_p A_p}{h_p}\rho_p A_p h_p} \\ &= \frac{c_t}{2E_t}\frac{A_p}{A_t}\sqrt{E_p \rho_p} \end{aligned} \tag{25}$$

Recalling the definition of the acoustic impedance, $z = \sqrt{\rho M}$ and the sonic wave speed, $c = \sqrt{M/\rho}$, and the longitudinal modulus, $M = B + \frac{4}{3}\mu = \frac{(1-\nu)}{(1+\nu)(1-2\nu)}E = f(\nu)E$. B is the bulk modulus, μ is the shear modulus and ν is the Poisson's ratio. We can re-write Eq.(25) as

$$\begin{aligned} \gamma &= \frac{A_p}{2A_t}\frac{\sqrt{M_t/\rho_t}}{E_t}\sqrt{\rho_f E_p} \\ &= \frac{A_p}{2A_t}\frac{1-\nu_t}{(1+\nu_t)(1-2\nu_t)}\frac{\sqrt{M_t/\rho_t}}{M_t}\sqrt{\rho_f E_p} \\ &= \frac{A_p}{2A_t}\frac{1-\nu_t}{(1+\nu_t)(1-2\nu_t)}\sqrt{\frac{(1+\nu_f)(1-2\nu_p)}{1-\nu_p}}\frac{\sqrt{\rho_f M_p}}{\sqrt{\rho_t M_t}} \\ &= \frac{A_p}{2A_t}\left(\frac{1-\nu_t}{(1+\nu_t)(1-2\nu_t)}\sqrt{\frac{(1+\nu_p)(1-2\nu_p)}{1-\nu_p}}\right)\frac{z_p}{z_t} \\ &= \frac{A_p}{2A_t}\frac{f(\nu_t)}{\sqrt{f(\nu_p)}}\frac{z_p}{z_t} \end{aligned} \tag{26}$$

This shows that γ is not just a function of the acoustic impedance ratio of the DNMM and the target, thus controlled by the density and elastic modulus of the DNMM, for a given target, but also related to the Poisson's ratio of the components. Specifically, $f(\nu)$ is asymptotic at the physical limits of ν, which has a significant impact on γ as shown below.

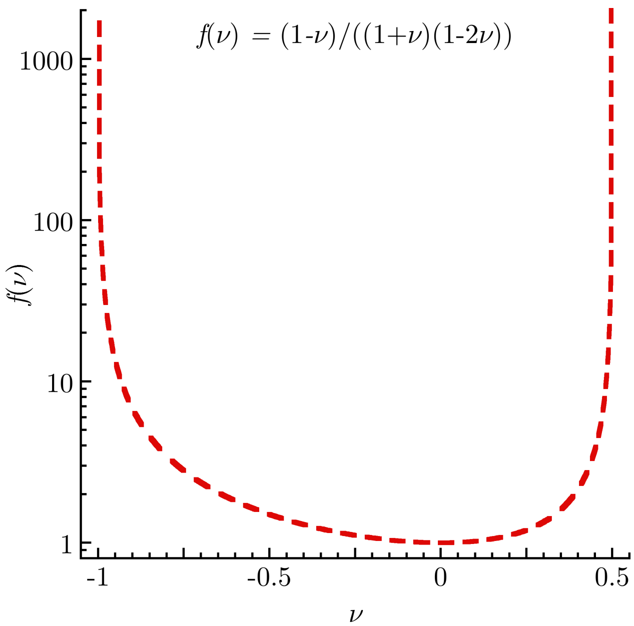

Figure 7: $f(\nu)$ vs. ν

We can simplify Eq.(26) by assuming that the load plate is made of steel with $\nu_t \approx 0.3$,

$$\gamma \approx \frac{A_p}{2A_t}\left(1.35\sqrt{\frac{(1+\nu_p)(1-2\nu_p)}{1-\nu_p}}\right)\frac{z_p}{z_t} \tag{27}$$

Naturally, γ is highly affected by the Poisson's ratio of the protective material and how the acoustic impedance trends with Poisson's ratio. The relation between ν and z_p is going to depend on the class of materials in question and will be the focus of a future study.

References

(1) Argatov, I.; Jokinen, M. *International Journal of Solids and Structures* **2013**, *50*, 3960–3966.