Supplementary Material for

Coexistence of ergodicity and nonergodicity in the aging two-state random walks

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I. STATISTICAL PROPERTIES OF THE TSRW MODEL

In the context of the fractal renewal theory, there have been studies on the statistical properties of the stochastic processes [1–3]. For the convenience of readers and discussions of other sections, by calling some known results presented in Refs. [1, 3], in this appendix we give specific discussions on the statistical properties of the TSRW model which are distinctly different from other stochastic processes.

A. Number of renewals between time 0 and t

The probability of n transitions (including jumping from the CTRW state to the LW state and jumping from the LW state to the CTRW state) between time 0 and t is

$$P_t(n) = P(0, t, n) = \langle \theta(t_n < t < t_{n+1}) \rangle, \tag{S1}$$

in which $\theta(t_n < t < t_{n+1})$ is 1 if the event in the parenthesis occurs and 0 otherwise, which implies that after n transitions, the evolving time t should be $t_n < t < t_{n+1}$, see Fig. 1 (c) of the main text. The Laplace form of Eq. (S1) with respect to t is

$$P_s(n) = \left\langle \int_0^\infty \theta(t_n < t < t_{n+1}) e^{-st} dt \right\rangle = \left\langle \int_{t_n}^{t_{n+1}} e^{-st} dt \right\rangle = \left\langle e^{-st_n} \frac{1 - e^{-s\tau_{n+1}}}{s} \right\rangle. \tag{S2}$$

Considering the fact that the first transition of the TSRW model is jumping from the CTRW state towards the LW state, by making use of $t_n = \sum_{i=1}^{n} \tau_i$, we have

$$P_s(n) = [\omega_r(s)\omega_j(s)]^{\frac{n}{2}} \frac{1 - \omega_r(s)}{s}, \tag{S3}$$

for even n, and

$$P_s(n) = [\omega_r(s)]^{\frac{n+1}{2}} [\omega_j(s)]^{\frac{n-1}{2}} \frac{1 - \omega_j(s)}{s}, \tag{S4}$$

for odd n. Eqs. (S3) and (S4) imply that for even transitions, the number of jumping from the CTRW state to the LW state n_r and the number of jumping from the LW state to the CTRW state n_j are both $\frac{n}{2}$, and for odd transitions, $n_r = \frac{n+1}{2}$ and $n_j = \frac{n-1}{2}$.

The average number of transitions in Laplace space can be obtained by making use of Eqs. (S3) and (S4), which is

$$\langle n(s) \rangle = \sum_{n=0}^{\infty} n P_s(n) = \frac{\omega_r(s)[1 + \omega_j(s)]}{s[1 - \omega_r(s)\omega_j(s)]}. \tag{S5}$$

Inverse transforming Eq. (S5) into the time domain, the average number of transitions is obtained, while, the specific value of α and β should be taken into account. After inserting Eqs. (12) and (13) into Eq. (S5), and performing the inverse Laplace transform, the long time behavior of $\langle n(t)\rangle$ is

$$\langle n(0,t)\rangle = \langle n(t)\rangle \simeq \begin{cases} \dfrac{2}{T_\alpha + T_\beta} t, & \text{case 1},\\ \dfrac{2}{\Gamma(1+\beta)\tau_\beta} t^\beta, & \text{case 2},\\ \dfrac{2}{\Gamma(1+\alpha)\tau_\alpha} t^\alpha, & \text{case 3}. \end{cases} \tag{S6}$$

From Eq. (S6), it can be seen that for both odd and even n, the average number of jumping from the CTRW state to the LW state $\langle n_r(t)\rangle$ is approximately equivalent to $\frac{\langle n(t)\rangle}{2}$ for long time limit,

$$\langle n_r(0,t)\rangle = \langle n_r(t)\rangle \simeq \begin{cases} \dfrac{1}{T_\alpha + T_\beta} t, & \text{case 1},\\ \dfrac{1}{\Gamma(1+\beta)\tau_\beta} t^\beta, & \text{case 2},\\ \dfrac{1}{\Gamma(1+\alpha)\tau_\alpha} t^\alpha, & \text{case 3}. \end{cases} \tag{S7}$$

B. The joint PDF of the forward recurrence time

The time $E = t_{N+1} - t$ is called the forward recurrence time. If t_N and t_{N+1} are defined by $t_N < t < t_{N+1}$, then N is also a random variable, see Fig. 1 of the main text. Define $f(t,E)$ to be the PDF of the forward recurrence time E, it can be calculated by summing over N on the PDF $f(t,E,N)$,

$$f(t,E,N) = \langle \delta((E - t_{N+1} + t)\theta(t_N < t < t_{N+1}))\rangle. \tag{S8}$$

Double Laplace transforming Eq. (S8) with respect to $t \to s$ and $E \to u$ yields

$$\begin{aligned} f(s,u,N) &= \mathscr{L}_{t,E}\Big\{f(t,E,N)\Big\} = \Big\langle \int_{t_N}^{t_{N+1}} dt \int_0^\infty dE e^{-st-uE}\delta(E - t_{N+1} + t)\Big\rangle \\ &= \Big\langle e^{-ut_{N+1}} \frac{e^{-(s-u)t_{N+1}} - e^{-(s-u)t_N}}{u-s}\Big\rangle. \end{aligned} \tag{S9}$$

For even N, it is

$$f(s,u,N) = \frac{[\omega_r(s)\omega_j(s)]^{\frac{N}{2}}[\omega_r(s) - \omega_r(u)]}{u-s}, \tag{S10}$$

in which

$$\begin{aligned}
&\langle e^{-st_{N+1}}\rangle = [\omega_r(s)]^{\frac{N}{2}+1}[\omega_j(s)]^{\frac{N}{2}},\\
&\langle e^{-st_N}\rangle = [\omega_r(s)\omega_j(s)]^{\frac{N}{2}},\\
&\langle e^{-u(t_{N+1}-t_N)}\rangle = \omega_r(u),
\end{aligned} \tag{S11}$$

are applied. Similarly, for odd N, it is

$$f(s,u,N) = \frac{[\omega_r(s)\omega_j(s)]^{\frac{N-1}{2}}[\omega_j(s)-\omega_j(u)]\omega_r(s)}{u-s}. \tag{S12}$$

After summing over N on $f(s,u,N)$, we have

$$f(s,u) = \frac{[\omega_r(s)-\omega_r(u)]+\omega_r(s)[\omega_j(s)-\omega_j(u)]}{(u-s)[1-\omega_r(s)\omega_j(s)]}, \tag{S13}$$

which is the expression of $f(t,E)$ in double Laplace space [1]. If $\omega_r(s)=\omega_j(s)=\omega(s)$, it reduces to the result for the single-state process [3],

$$f(s,u) = \frac{\omega(s)-\omega(u)}{(u-s)[1-\omega(s)]}. \tag{S14}$$

It is important to calculate the joint PDFs of the forward recurrence time E provided that the particle is located in the CTRW state or in the LW state at time t, here we define them as $f_r(t,E)$ and $f_j(t,E)$, of which the double Laplace form are $f_r(s,u)$ and $f_j(s,u)$ respectively. By making use of Eq. (S10), we have

$$f_r(s,u) = \sum_{N=0,\ N\ \text{even}}^{\infty} f(s,u,N) = \frac{\omega_r(s)-\omega_r(u)}{(u-s)[1-\omega_r(s)\omega_j(s)]}, \tag{S15}$$

and by making use of Eq. (S12), we have

$$f_j(s,u) = \sum_{N=0,\ N\ \text{odd}}^{\infty} f(s,u,N) = \frac{\omega_r(s)[\omega_j(s)-\omega_j(u)]}{(u-s)[1-\omega_r(s)\omega_j(s)]}. \tag{S16}$$

C. The persistence probability

Define $P_n(t,t+\tau)$ to be the probability of n transitions between time t and $t+\tau$. The probability of no renewal happening between time t and $t+\tau$, i.e., the persistence probability, is the one we care. Define $P_{r,r,0}(t,t+\tau)$ to be the persistence probability that the particle is located in the CTRW state at time t and no renewal happens between time t and $t+\tau$, it can be calculated by

$$P_{r,r,0}(t,t+\tau) = \int_\tau^\infty f_r(t,E)dE. \tag{S17}$$

Define $P_{j,j,0}(t, t+\tau)$ to be the persistence probability that the particle is located in the LW state at time t and no renewal happens between time t and $t+\tau$, which is

$$P_{j,j,0}(t, t+\tau) = \int_{\tau}^{\infty} f_j(t, E) dE. \tag{S18}$$

Double Laplace transforming Eq. (S17) with respect to $t \to s$ and $E \to u$ yields

$$P_{r,r,0}(s, u) = \frac{1 - s f_r(s, u)}{us}, \tag{S19}$$

and after inserting Eq. (S15) into Eq. (S19), it arrives at

$$P_{r,r,0}(s, u) = \frac{(u-s)[1 - \omega_r(s)\omega_j(s)] - s[\omega_r(s) - \omega_r(u)]}{us(u-s)[1 - \omega_r(s)\omega_j(s)]}, \tag{S20}$$

which is the expression of the persistence probability $P_{r,r,0}(t, t+\tau)$ in double Laplace space.

Similarly, double Laplace transforming Eq. (S18) with respect to $t \to s$ and $E \to u$ yields

$$P_{j,j,0}(s, u) = \frac{1 - s f_j(s, u)}{us}, \tag{S21}$$

and after inserting Eq. (S16) into Eq. (S21), it arrives at

$$P_{j,j,0}(s, u) = \frac{u - s - u\omega_r(s)\omega_j(s) + s\omega_r(s)\omega_j(u)}{us(u-s)[1 - \omega_r(s)\omega_j(s)]}, \tag{S22}$$

which is the expression of the persistence probability $P_{j,j,0}(t, t+\tau)$ in double Laplace space.

The persistence probability can also be obtained through a second way. Take $P_{j,j,0}(t, t+\tau)$ for example, define $P_{j,0}(t, t+\tau)$ to be the probability that no renewal happens between time t and $t+\tau$ provided that the particle is located in the LW state, then the persistence probability can be expressed as

$$P_{j,j,0}(t, t+\tau) = P_j(t) P_{j,0}(t, t+\tau), \tag{S23}$$

where $P_j(t)$ is given by Eq. (6). $P_{j,0}(t, t+\tau)$ can be calculated by [3]

$$P_{j,0}(s, u) = \frac{1 - s f(s, u)}{us}, \tag{S24}$$

in which $f(s, u)$ is given by Eq. (S14). For $0 < \beta < 1$, it is

$$P_{j,0}(s, u) \simeq \frac{us^{\beta} - su^{\beta}}{us(u-s)s^{\beta}}, \tag{S25}$$

and for $1 < \beta < 2$, it is

$$P_{j,0}(s, u) \simeq \frac{\tau_\beta}{T_\beta} \frac{u^{\beta-1} - s^{\beta-1}}{s(u-s)}. \tag{S26}$$

Inverse Laplace transforming Eqs. (S25) and (S26) yields

$$P_{j,0}(t, t+\tau) \;=\; \frac{\sin(\pi\beta)}{\pi} B\Big(\frac{1}{1+\tau/t}; \beta, 1-\beta\Big), \tag{S27}$$

for $0 < \beta < 1$, and

$$P_{j,0}(t, t+\tau) \;=\; \frac{\tau_\beta}{T_\beta} \frac{1}{\Gamma(2-\beta)} [\tau^{1-\beta} - (t+\tau)^{1-\beta}], \tag{S28}$$

for $1 < \beta < 2$. Here

$$B(x; a, b) = \int_o^x t^{a-1}(1-t)^{b-1} dt, \tag{S29}$$

is the incomplete β function. After inserting Eqs. (10), (S27), and (S28) into Eq. (S23), the persistence probability $P_{j,j,0}(t, t+\tau)$ is obtained.

II. MSDS OF THE CTRW STATE

To calculate the EAMSD of the CTRW state, the specific value of α and β should be taken into account, i.e., Eq. (S7). After inserting Eq. (S7) into Eq. (18), for the weak aging case $t_a \ll t$, we have

$$\langle \Delta x_{t_a}^2(t) \rangle_{\mathrm{CTRW}} \simeq \begin{cases} \dfrac{\sigma^2}{T_\alpha + T_\beta} t, & \text{case 1,} \\ \dfrac{\sigma^2}{\Gamma(1+\beta)\tau_\beta} t^\beta, & \text{case 2,} \\ \dfrac{\sigma^2}{\Gamma(1+\alpha)\tau_\alpha} t^\alpha, & \text{case 3,} \end{cases} \tag{S30}$$

and for the strong aging case $t_a \gg t$, we have

$$\langle \Delta x_{t_a}^2(t) \rangle_{\mathrm{CTRW}} \simeq \begin{cases} \dfrac{\sigma^2}{T_\alpha + T_\beta} t, & \text{case 1,} \\ \dfrac{\sigma^2}{\Gamma(\beta)\tau_\beta} t_a^{\beta-1} t, & \text{case 2,} \\ \dfrac{\sigma^2}{\Gamma(\alpha)\tau_\alpha} t_a^{\alpha-1} t, & \text{case 3.} \end{cases} \tag{S31}$$

To calculate the ensemble averaged TAMSD Eq. (19), the specific value of α and β still should be taken into account. For case 3, after inserting Eq. (S7) into Eq. (19), we have

$$\begin{aligned} \langle \overline{\delta_{t_a}^2(\Delta, T)} \rangle_{\mathrm{CTRW}} \;&=\; \frac{\sigma^2}{\Gamma(2+\alpha)\tau_\alpha} \Big[(t_a+T)^{1+\alpha} - (t_a+T-\Delta)^{1+\alpha} - (t_a+\Delta)^{1+\alpha} + t_a^{1+\alpha} \Big] \\ &\simeq\; \frac{\sigma^2}{\Gamma(1+\alpha)} \frac{\Delta}{T^{1-\alpha}} \Lambda_\alpha\Big(\frac{t_a}{T}\Big), \end{aligned} \tag{S32}$$

in which $\Lambda_\alpha(\frac{t_a}{T}) = (1+\frac{t_a}{T})^\alpha - (\frac{t_a}{T})^\alpha$. For the weak aging case $t_a \ll T$, from Eq. (S32) we have

$$\langle\overline{\delta^2_{t_a}(\Delta,T)}\rangle_{\mathrm{CTRW}} \simeq \frac{\sigma^2}{\Gamma(1+\alpha)\tau_\alpha}\frac{\Delta}{T^{1-\alpha}}. \tag{S33}$$

For the strong aging case $t_a \gg T$, from Eq. (S32) we have

$$\langle\overline{\delta^2_{t_a}(\Delta,T)}\rangle_{\mathrm{CTRW}} \simeq \frac{\sigma^2}{\Gamma(\alpha)\tau_\alpha}t_a^{\alpha-1}\Delta. \tag{S34}$$

The calculations on Eq. (19) for case 2 can be performed in a similar way. For case 1, from Eq. (S7) it can be obtained that $\langle n_r(t_a, t_a+t)\rangle = \langle n_r(0,t)\rangle = t/(T_\alpha+T_\beta)$, which is independent from the aging time t_a, and correspondingly,

$$\langle\overline{\delta^2_{t_a}(\Delta,T)}\rangle_{\mathrm{CTRW}} = \frac{\sigma^2}{T_\alpha+T_\beta}\Delta. \tag{S35}$$

Based on the above calculations, we have the ensemble averaged TAMSD of the CTRW state. For the weak aging case $t_a \ll T$,

$$\langle\overline{\delta^2_{t_a}(\Delta,T)}\rangle_{\mathrm{CTRW}} \simeq \begin{cases} \dfrac{\sigma^2}{T_\alpha+T_\beta}\Delta, & \text{case 1,} \\ \dfrac{\sigma^2}{\Gamma(1+\beta)\tau_\beta}T^{\beta-1}\Delta, & \text{case 2,} \\ \dfrac{\sigma^2}{\Gamma(1+\alpha)\tau_\alpha}T^{\alpha-1}\Delta, & \text{case 3,} \end{cases} \tag{S36}$$

and for the strong aging case $t_a \gg T$,

$$\langle\overline{\delta^2_{t_a}(\Delta,T)}\rangle_{\mathrm{CTRW}} \simeq \begin{cases} \dfrac{\sigma^2}{T_\alpha+T_\beta}\Delta, & \text{case 1,} \\ \dfrac{\sigma^2}{\Gamma(\beta)\tau_\beta}t_a^{\beta-1}\Delta, & \text{case 2,} \\ \dfrac{\sigma^2}{\Gamma(\alpha)\tau_\alpha}t_a^{\alpha-1}\Delta, & \text{case 3.} \end{cases} \tag{S37}$$

III. MSDS OF THE LW STATE

The scaling Green-Kubo relation is a generalized formulation that is valid for systems with long-range or non-stationary correlations for which the standard approach is no longer valid [4, 5]. It is defined by using the velocity correlation function,

$$C_v(t,t+\tau) = \langle v(t)v(t+\tau)\rangle \simeq Ct^{\nu-2}\phi(\frac{\tau}{t}), \tag{S38}$$

in which $C > 0$ is a constant, $\nu > 1$ is the exponent, and $\phi(z)$ is a scaling function which is limited by two power laws,

$$\begin{aligned} \phi(z) &< c_1 z^{-\delta_1} \quad \text{with } 2-\nu \le \delta_1 < 1 \text{ for } z \to 0, \\ \phi(z) &< c_u z^{-\delta_u} \quad \text{with } \delta_u > 1-\nu \text{ for } z \to \infty, \end{aligned} \tag{S39}$$

in which c_1 and c_u are positive constants.

For the LW state, The EAMSD can be calculated by making use of the scaling Green-Kubo relation,

$$\begin{aligned} \langle x^2(t)\rangle_{\mathrm{LW}} &= \int_0^t dt_2 \int_0^t dt_1 C_\nu(t_1,t_2) \\ &\simeq 2C\int_0^t dt_2 t_2^{\nu-1}\int_0^\infty (1+z)^{-\nu}\phi(z)dz \\ &= \frac{2C}{\nu}t^\nu \int_0^\infty (1+z)^{-\nu}\phi(z)dz, \end{aligned} \tag{S40}$$

in which $z = \frac{t_2-t_1}{t_1}$ is applied. Eq. (S40) can also be expressed as

$$\langle x^2(t)\rangle_{\mathrm{LW}} = 2D_\nu t^\nu, \tag{S41}$$

with

$$D_\nu = \frac{C}{\nu}\int_0^\infty (1+z)^{-\nu}\phi(z)dz. \tag{S42}$$

After considering aging, the EAMSD becomes

$$\begin{aligned} \langle \Delta x_{t_a}^2(t)\rangle_{\mathrm{LW}} &= \left\langle \left[x(t_a+t)-x(t_a)\right]^2 \right\rangle \\ &= \int_{t_a}^{t_a+t} dt_2 \int_{t_a}^{t_a+t} dt_1 C_\nu(t_1,t_2) \\ &\simeq 2C\int_0^t dt_2 \int_0^{t_2} dt_1 (t_1+t_a)^{\nu-2}\phi(\frac{t_2-t_1}{t_1+t_a}). \end{aligned} \tag{S43}$$

For the weak aging case $t_a \ll t$, after omitting t_a and introducing $z = \frac{t_2-t_1}{t_1}$, it is

$$\begin{aligned} \langle \Delta x_{t_a}^2(t)\rangle_{\mathrm{LW}} &\simeq 2C\int_0^t dt_2 t_2^{\nu-1}\int_0^\infty (1+z)^{-\nu}\phi(z)dz \\ &= 2D_\nu t^\nu, \end{aligned} \tag{S44}$$

with

$$D_\nu = \frac{C}{\nu}\int_0^\infty (1+z)^{-\nu}\phi(z)dz. \tag{S45}$$

For the strong aging case $t_a \gg t$, Eq. (S43) can be approximately expressed as

$$\begin{aligned} \langle \Delta x_{t_a}^2(t) \rangle_{\rm LW} &\simeq 2C \int_0^t dt_2 t_2^{\nu-1} \int_0^{t_2} dt_1 t_a^{\nu-2} \phi(\frac{t_2 - t_1}{t_a}) \\ &\simeq \frac{2c_1 C}{(\nu - q - 1)(\nu - q)} t_a^q t^{\nu-q}, \end{aligned} \tag{S46}$$

in which $\phi(z) \simeq c_1 z^{-\delta_1}$ for small z with $\delta_1 = 2 - \nu - q$, and q is the exponent of the variance of the velocity $\langle v^2(t) \rangle \sim t^q$ with $-1 \le q < \nu - 1$.

The ensemble averaged TAMSD of the LW state Eq. (21) can be calculated in the following way. Considering $\Delta \ll T$, in fact, the leading term of the integral in Eq. (21) is contributed by the part with $\Delta \ll t$ no matter what the aging time t_a is. Hence, utilizing the result of $\langle \Delta x_{t_a}^2(t) \rangle$ for the strong aging case Eq. (S46), Eq. (21) can be approximately expressed as

$$\begin{aligned} \langle \overline{\delta_{t_a}^2(\Delta, T)} \rangle_{\rm LW} &\simeq \frac{1}{T - \Delta} \int_{t_a}^{t_a + T - \Delta} \frac{2c_1 C}{(\nu - q - 1)(\nu - q)} t^q \Delta^{\nu-q} dt \\ &\simeq \frac{2c_1 C}{(\nu - q - 1)(\nu - q)} T^q \Delta^{\nu-q} [(1 + \frac{t_a}{T})^{1+q} - (\frac{t_a}{T})^{1+q}]. \end{aligned} \tag{S47}$$

For the weak aging case $t_a \ll T$, it is

$$\langle \overline{\delta_{t_a}^2(\Delta, T)} \rangle_{\rm LW} \simeq \frac{2c_1 C}{(\nu - q - 1)(\nu - q)(1 + q)} T^q \Delta^{\nu-q}, \tag{S48}$$

and for the strong aging case $t_a \gg T$, it is

$$\langle \overline{\delta_{t_a}^2(\Delta, T)} \rangle_{\rm LW} \simeq \frac{2c_1 C}{(\nu - q - 1)(\nu - q)} t_a^q \Delta^{\nu-q}. \tag{S49}$$

Eqs. (S44), (S46), (S48), and (S49) are the generic expressions of the MSDs of the LW state. To obtain the specific results, the specific value of the power exponent α and β should be taken into account. The detailed calculations and the specific results are presented as follows.

A. case 1, $1 < \min\{\alpha, \beta\} < 2$

After inserting Eqs. (12) and (13) into Eq. (S22), the persistence probability can be approximately expressed as

$$P_{j,j,0}(s, u) \simeq \frac{\tau_\beta}{T_\alpha + T_\beta} \frac{u^{\beta-1} - s^{\beta-1}}{s(u - s)}, \tag{S50}$$

for long time limit. Compared Eq. (S50) with Eq. (S26), we have

$$\begin{aligned} P_{j,j,0}(t, t + \tau) &= \frac{\tau_\beta}{T_\alpha + T_\beta} \frac{1}{\Gamma(2 - \beta)} [\tau^{1-\beta} - (t + \tau)^{1-\beta}] \\ &= \frac{\tau_\beta}{T_\alpha + T_\beta} \frac{1}{\Gamma(2 - \beta)} t^{1-\beta} [(\frac{\tau}{t})^{1-\beta} - (1 + \frac{\tau}{t})^{1-\beta}]. \end{aligned} \tag{S51}$$

After inserting Eq. (S51) into Eq. (23), the velocity correlation function is

$$\langle v(t)v(t+\tau)\rangle = v^2 \frac{\tau_\beta}{T_\alpha + T_\beta} \frac{1}{\Gamma(2-\beta)} t^{1-\beta}[(\frac{\tau}{t})^{1-\beta} - (1+\frac{\tau}{t})^{1-\beta}]. \tag{S52}$$

Compared Eq. (S52) with the scaling Green-Kubo relation Eq. (S38), it can be seen that $\nu = 3-\beta$, $C = v^2 \frac{\tau_\beta}{T_\alpha+T_\beta} \frac{1}{\Gamma(2-\beta)}$, $\phi(z) = [z^{1-\beta} - (1+z)^{1-\beta}]$, and $q = 0$. Besides, since $\phi(z) \simeq c_1 z^{-\delta_1}$ for small z with $\delta_1 = 2 - \nu - q$, we also have $\delta_1 = \beta - 1$ and $c_1 = 1$ for small z. After substituting them into Eqs. (S44), (S46), (S48), and (S49), we have

$$\langle \Delta x_{t_a}^2(t)\rangle_{\mathrm{LW}} \simeq \begin{cases} \dfrac{2(\beta-1)\tau_\beta v^2}{\Gamma(4-\beta)(T_\alpha+T_\beta)} t^{3-\beta}, & t_a \ll t, \\ \dfrac{2\tau_\beta v^2}{\Gamma(4-\beta)(T_\alpha+T_\beta)} t^{3-\beta}, & t_a \gg t, \end{cases} \tag{S53}$$

and

$$\langle \overline{\delta_{t_a}^2(\Delta, T)}\rangle_{\mathrm{LW}} \simeq \begin{cases} \dfrac{2\tau_\beta v^2}{\Gamma(4-\beta)(T_\alpha+T_\beta)} \Delta^{3-\beta}, & t_a \ll T, \\ \dfrac{2\tau_\beta v^2}{\Gamma(4-\beta)(T_\alpha+T_\beta)} \Delta^{3-\beta}, & t_a \gg T. \end{cases} \tag{S54}$$

Note that, in calculating Eq. (S53), the result

$$\begin{aligned} D_\nu &= \frac{C}{\nu}\int_0^\infty (1+z)^{-\nu}\phi(z)dz \\ &= \frac{\tau_\beta v^2}{(T_\alpha+T_\beta)\Gamma(2-\beta)(3-\beta)} \int_0^\infty (1+z)^{\beta-3}[z^{1-\beta} - (1+z)^{1-\beta}]dz \\ &= \frac{\tau_\beta v^2(2-\beta)}{(T_\alpha+T_\beta)\Gamma(4-\beta)} \Big[\int_0^\infty z^{1-\beta}(1+z)^{\beta-3}dz - \int_0^\infty (1+z)^{-2}dz\Big] \\ &= \frac{(\beta-1)\tau_\beta v^2}{\Gamma(4-\beta)(T_\alpha+T_\beta)}. \end{aligned} \tag{S55}$$

is applied.

B. case 2, $0 < \beta < 1$ and $\alpha > \beta$

After inserting Eqs. (12) and (13) into Eq. (S22), the persistence probability can be approximately expressed as

$$P_{j,j,0}(s,u) \simeq \frac{us^\beta - su^\beta}{us(u-s)s^\beta}, \tag{S56}$$

for long time limit. Compared Eq. (S56) with Eq. (S25), we have

$$P_{j,j,0}(t,t+\tau) = \frac{\sin(\pi\beta)}{\pi} B\Big(\frac{1}{1+\tau/t};\beta, 1-\beta\Big). \tag{S57}$$

After inserting Eq. (S57) into Eq. (23), the velocity correlation function is

$$\langle v(t)v(t+\tau)\rangle = v^2 \frac{\sin(\pi\beta)}{\pi} B\Big(\frac{1}{1+\tau/t};\beta, 1-\beta\Big). \tag{S58}$$

Compared Eq. (S58) with the scaling Green-Kubo relation Eq. (S38), it can be seen that $\nu = 2$, $C = \frac{v^2\sin(\pi\beta)}{\pi}$, $\phi(z) = B\big(\frac{1}{1+z};\beta, 1-\beta\big)$, and $q = 0$. Besides, since $\phi(z) \simeq c_1 z^{-\delta_1}$ for small z with $\delta_1 = 2-\nu-q$, we also have $\delta_1 = 0$ and $c_1 = B\big(\frac{1}{1+z};\beta, 1-\beta\big) \simeq \frac{\pi}{\sin(\pi\beta)}$ for small z. After substituting them into Eqs. (S44), (S46), (S48), and (S49), we have

$$\langle \Delta x_{t_a}^2(t)\rangle_{\mathrm{LW}} \simeq \begin{cases} v^2(1-\beta)t^2, & t_a \ll t, \\ v^2t^2, & t_a \gg t, \end{cases} \tag{S59}$$

and

$$\langle \overline{\delta_{t_a}^2(\Delta, T)}\rangle_{\mathrm{LW}} \simeq \begin{cases} v^2\Delta^2, & t_a \ll T, \\ v^2\Delta^2, & t_a \gg T. \end{cases} \tag{S60}$$

Note that, in calculating Eq. (S59), with introducing $z' = \frac{1}{1+z}$, the result

$$\begin{aligned} D_\nu &= \frac{C}{\nu}\int_0^\infty (1+z)^{-\nu}\phi(z)dz \\ &= \frac{v^2\sin(\pi\beta)}{2\pi}\int_0^\infty (1+z)^{-2} B\Big(\frac{1}{1+z};\beta, 1-\beta\Big)dz \\ &\simeq \frac{v^2\sin(\pi\beta)}{2\pi}\int_0^1 B\big(z';\beta, 1-\beta\big)dz' \\ &= \frac{(1-\beta)v^2}{2}, \end{aligned} \tag{S61}$$

is applied.

C. case 3, $0 < \alpha < 1$ and $\alpha < \beta$

After inserting Eqs. (12) and (13) into Eq. (S22), it can be seen that $P_{j,j,0}(t,t+\tau) = 0$ for long time limit. This result is actually related to the state occupation mechanism (see Section III of the main text), i.e., after inserting Eqs. (12) and (13) into Eqs. (9) and (10), we have $P_j(t) = 0$ for long time limit, which implies that the probability of finding the particle in the LW state is 0, and not surprisingly $P_{j,j,0}(t, t+\tau) = 0$. Hence, different from case 1 and 2, for case 3, $P_{j,j,0}(t, t+\tau)$ can not be directly obtained from Eq. (S22). Instead, using Eq. (S23) is a compromise way to acquire $P_{j,j,0}(t, t+\tau)$ at intermediate timescales. The following discussions are divided into case 3a ($0 < \alpha < \beta < 1$) and case 3b ($0 < \alpha < 1 < \beta < 2$).

1. *case 3a,* $0 < \alpha < \beta < 1$

After inserting Eqs. (12) and (13) into Eq. (10), we have

$$P_j(s) \simeq \frac{\tau_\beta}{\tau_\alpha} s^{-(1+\alpha-\beta)}, \tag{S62}$$

and

$$P_j(t) = \frac{\tau_\beta}{\Gamma(1+\alpha-\beta)\tau_\alpha} t^{\alpha-\beta}, \tag{S63}$$

which approaches to 0 for long time limit just as discussed above. For intermediate timescales, inserting Eqs. (S27) and (S63) into Eq. (S23) yields

$$\begin{aligned} \langle v(t)v(t+\tau)\rangle &= v^2 P_j(t) P_{j,0}(t, t+\tau) \\ &= v^2 \frac{\tau_\beta}{\Gamma(1+\alpha-\beta)\tau_\alpha} t^{\alpha-\beta} \frac{\sin(\pi\beta)}{\pi} B\Big(\frac{1}{1+\tau/t}; \beta, 1-\beta\Big). \end{aligned} \tag{S64}$$

Compared Eq. (S64) with the scaling Green-Kubo relation Eq. (S38), it can be seen that $\nu = 2+\alpha-\beta$, $C = v^2 \frac{\tau_\beta}{\Gamma(1+\alpha-\beta)\tau_\alpha} \frac{\sin(\pi\beta)}{\pi}$, $\phi(z) = B\Big(\frac{1}{1+z}; \beta, 1-\beta\Big)$, and $q = \alpha-\beta$. Besides, since $\phi(z) \simeq c_1 z^{-\delta_1}$ for small z with $\delta_1 = 2-\nu-q$, we also have $\delta_1 = 0$ and $c_1 = B\Big(\frac{1}{1+z}; \beta, 1-\beta\Big) \simeq \frac{\pi}{\sin(\pi\beta)}$ for small z. After substituting them into Eqs. (S44), (S46), (S48), and (S49), we have

$$\langle \Delta x_{t_a}^2(t)\rangle_{\mathrm{LW}} \simeq \begin{cases} \dfrac{\tau_\beta v^2}{\Gamma(3+\alpha-\beta)\tau_\alpha}\Big[1 - \dfrac{\Gamma(1+\alpha)}{\Gamma(2+\alpha-\beta)\Gamma(\beta)}\Big] t^{2+\alpha-\beta}, & t_a \ll t, \\ \dfrac{\tau_\beta v^2}{\Gamma(1+\alpha-\beta)\tau_\alpha} t_a^{\alpha-\beta} t^2, & t_a \gg t, \end{cases} \tag{S65}$$

and

$$\langle \overline{\delta_{t_a}^2(\Delta, T)}\rangle_{\mathrm{LW}} \simeq \begin{cases} \dfrac{\tau_\beta v^2}{\Gamma(2+\alpha-\beta)\tau_\alpha} T^{\alpha-\beta}\Delta^2, & t_a \ll T, \\ \dfrac{\tau_\beta v^2}{\Gamma(1+\alpha-\beta)\tau_\alpha} t_a^{\alpha-\beta}\Delta^2, & t_a \gg T. \end{cases} \tag{S66}$$

Note that, in calculating Eq. (S65), with introducing $z' = \frac{1}{1+z}$, the result

$$\begin{aligned} D_\nu &= \frac{C}{\nu}\int_0^\infty (1+z)^{-\nu}\phi(z)dz \\ &= \frac{\tau_\beta v^2}{\Gamma(1+\alpha-\beta)(2+\alpha-\beta)\tau_\alpha}\frac{\sin(\pi\beta)}{\pi}\int_0^\infty (1+z)^{-(2+\alpha-\beta)}B\Big(\frac{1}{1+z};\beta,1-\beta\Big)dz \\ &\simeq \frac{\tau_\beta v^2}{\Gamma(3+\alpha-\beta)\tau_\alpha}\frac{\sin(\pi\beta)}{\pi}\int_0^1 B\Big(z';\beta,1-\beta\Big)dz'^{1+\alpha-\beta} \\ &= \frac{\tau_\beta v^2}{\Gamma(3+\alpha-\beta)\tau_\alpha}\frac{\sin(\pi\beta)}{\pi}\Big[\frac{\pi}{\sin(\pi\beta)} - \int_0^1 z'^{\alpha}(1-z')^{-\beta}dz'\Big] \\ &= \frac{\tau_\beta v^2}{\Gamma(3+\alpha-\beta)\tau_\alpha}\Big[1 - \frac{\Gamma(1+\alpha)}{\Gamma(2+\alpha-\beta)\Gamma(\beta)}\Big] \end{aligned} \tag{S67}$$

is applied. Besides, considering $0 < \alpha < \beta < 1$, if α is treated as slightly smaller than β, Eq. (S67) can be approximately expressed as $D_\nu \simeq \frac{\tau_\beta v^2}{\Gamma(3+\alpha-\beta)\tau_\alpha}(1-\beta)$.

2. *case 3b,* $0 < \alpha < 1 < \beta < 2$

After inserting Eqs. (12) and (13) into Eq. (10), we have

$$P_j(s) \simeq \frac{T_\beta}{\tau_\alpha}s^{-\alpha}, \tag{S68}$$

and

$$P_j(t) = \frac{T_\beta}{\Gamma(\alpha)\tau_\alpha}t^{\alpha-1}, \tag{S69}$$

which still approaches to 0 for long time limit. For intermediate timescales, inserting Eqs. (S28) and (S69) into Eq. (S23) yields

$$\begin{aligned} \langle v(t)v(t+\tau)\rangle &= v^2 P_j(t)P_{j,0}(t,t+\tau) \\ &= v^2\frac{T_\beta}{\Gamma(\alpha)\tau_\alpha}t^{\alpha-1}\frac{\tau_\beta}{T_\beta}\frac{1}{\Gamma(2-\beta)}[\tau^{1-\beta} - (t+\tau)^{1-\beta}] \\ &= \frac{\tau_\beta v^2}{\Gamma(\alpha)\Gamma(2-\beta)\tau_\alpha}t^{\alpha-\beta}\Big[\Big(\frac{\tau}{t}\Big)^{1-\beta} - \Big(1+\frac{\tau}{t}\Big)^{1-\beta}\Big]. \end{aligned} \tag{S70}$$

Compared Eq. (S70) with the scaling Green-Kubo relation Eq. (S38), it can be seen that $\nu = 2+\alpha-\beta$, $C = \frac{\tau_\beta v^2}{\Gamma(\alpha)\Gamma(2-\beta)\tau_\alpha}$, $\phi(z) = [z^{1-\beta} - (1+z)^{1-\beta}]$, and $q = \alpha - 1$. Besides, since $\phi(z) \simeq c_1 z^{-\delta_1}$ for small z with $\delta_1 = 2-\nu-q$, we also have $\delta_1 = \beta - 1$ and $c_1 = 1$ for small z. After substituting them into Eqs. (S44), (S46), (S48), and (S49), we have

$$\langle \Delta x_{t_a}^2(t)\rangle_{\mathrm{LW}} \simeq \begin{cases} \dfrac{2\tau_\beta v^2}{\tau_\alpha}\Big[\dfrac{1}{\Gamma(3+\alpha-\beta)} - \dfrac{1}{\Gamma(1+\alpha)\Gamma(2-\beta)}\Big]t^{2+\alpha-\beta}, & t_a \ll t, \\ \dfrac{2\tau_\beta v^2}{\Gamma(\alpha)\Gamma(4-\beta)\tau_\alpha}t_a^{\alpha-1}t^{3-\beta}, & t_a \gg t, \end{cases} \tag{S71}$$

and

$$\overline{\langle \delta^2_{t_a}(\Delta, T)\rangle}_{\mathrm{LW}} \simeq \begin{cases} \dfrac{2\tau_\beta v^2}{\Gamma(\alpha+1)\Gamma(4-\beta)\tau_\alpha} T^{\alpha-1}\Delta^{3-\beta}, & t_a \ll T, \\ \dfrac{2\tau_\beta v^2}{\Gamma(\alpha)\Gamma(4-\beta)\tau_\alpha} t_a^{\alpha-1}\Delta^{3-\beta}, & t_a \gg T. \end{cases} \tag{S72}$$

Note that, in calculating Eq. (S71), with introducing $z' = \frac{1}{1+z}$, the result

$$\begin{aligned} D_\nu &= \frac{C}{\nu}\int_0^\infty (1+z)^{-\nu}\phi(z)dz \\ &= \frac{\tau_\beta v^2}{(2+\alpha-\beta)\Gamma(\alpha)\Gamma(2-\beta)\tau_\alpha}\int_0^\infty (1+z)^{-(2+\alpha-\beta)}[z^{1-\beta}-(1+z)^{1-\beta}]dz \\ &= \frac{\tau_\beta v^2}{(2+\alpha-\beta)\Gamma(\alpha)\Gamma(2-\beta)\tau_\alpha}\left[\frac{\Gamma(\alpha)\Gamma(2-\beta)}{\Gamma(2+\alpha-\beta)} - \frac{1}{\alpha}\right] \\ &= \frac{\tau_\beta v^2}{\tau_\alpha}\left[\frac{1}{\Gamma(3+\alpha-\beta)} - \frac{1}{\Gamma(1+\alpha)\Gamma(2-\beta)}\right] \end{aligned} \tag{S73}$$

is applied.

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