# AFM-based spherical indentation of a brush-coated cell: Modeling the bottom effect

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## Supplementary Information

## A. Asymptotic constants for a bonded layer

The asymptotic constants $a_0$ and $a_1$ can be evaluated as follows (Alexandrov and Pozharskii, 2001):

$$a_m = \frac{(-1)^m}{[(2m)!!]^2} \int_0^\infty [1 - L(u)] u^{2m} \mathrm{d}u. \tag{A.1}$$

In the case of an isotropic elastic layer bonded to a rigid base, the kernel function is given by

$$L(u) = \frac{2\varkappa \sinh(2u) - 4u}{2\varkappa \cosh(2u) + 1 + \varkappa^2 + 4u^2}, \tag{A.2}$$

where $\varkappa = 3 - 4\nu$ is Kolosov's constant, and $\nu$ is Poisson's ratio.

The normalized asymptotic constants are defined as

$$\alpha_0 = \frac{8a_0}{3\pi}, \quad \alpha_1 = -\frac{8a_1}{3\pi}.$$

In view of (A.1), we have

$$\alpha_0 = \frac{8}{3\pi} \int_0^\infty [1 - L(u)] \, \mathrm{d}u, \quad \alpha_1 = \frac{2}{3\pi} \int_0^\infty [1 - L(u)] u^2 \mathrm{d}u, \tag{A.3}$$

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where $L(u)$ is given by (A.2).

Practically, the improper integrals (A.1) and (A.3) can be evaluated numerically by replacing the infinite upper limit of the integral by a finite upper limit, that is

$$a_m \approx \frac{(-1)^m}{[(2m)!!]^2} \int_0^M [1 - L(u)] u^{2m} \mathrm{d}u. \tag{A.4}$$

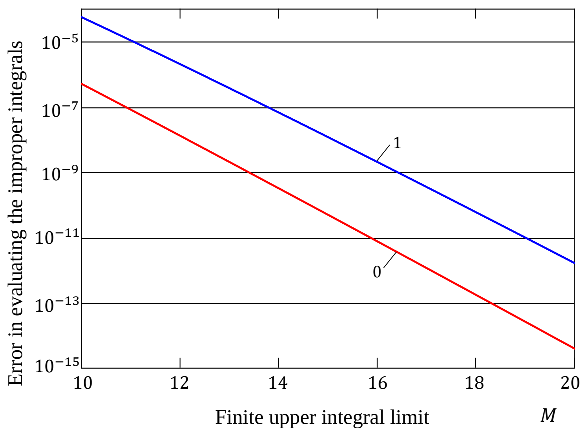


Figure 1: Accuracy of the numerical evaluation of the asymptotic constants $a_0$ (line 0) and $a_1$ (line 1), which is the same as that for $\alpha_0$ and $\alpha_1$.

The upper bound for the error of such approximation is illustrated in Fig. 1 for the case $\nu = 0.5$. The error is exponentially decaying with the increase of the upper limit $M$ in the definite integral (A.4).

# B. Accuracy of the approximate solutions

Let us introduce the notation

$$\varepsilon = \frac{a}{H}, \quad \varpi = \frac{\sqrt{R\delta}}{H}, \quad \tilde{F} = \frac{RF}{E^* H^3}, \quad \tilde{\delta} = \frac{R\delta}{H^2}. \tag{B.1}$$

The fourth-order asymptotic solution obtained by Vorovich et al. (1974) has the following form:

$$F = \frac{4}{3}\frac{E^*}{R} a^3 \left(1 - \varepsilon^3 \frac{8a_1}{3\pi}\right), \tag{B.2}$$

$$F = \frac{4}{3} E^* \delta a \left\{ 1 + \varepsilon \frac{4a_0}{3\pi} + \varepsilon^2 \left(\frac{4a_0}{3\pi}\right)^2 + \varepsilon^3 \left(\frac{4a_0}{3\pi}\right)^3 + \varepsilon^4 \left(\frac{4a_0}{3\pi}\right)^4 + \varepsilon^3 \frac{8a_1}{15\pi} \left(1 + \varepsilon \frac{8a_0}{3\pi}\right) \right\}. \tag{B.3}$$

Using the asymptotic solution (B.2), (B.3) and the dimensionless variables (B.1), we can represent the approximate force-displacement relation in the *parametric* form as

$$\tilde{F} = \frac{4}{3}\varepsilon^3\biggl(1 - \varepsilon^3\frac{8a_1}{3\pi}\biggr), \tag{B.4}$$

$$\tilde{\delta} = \frac{\varepsilon^2\biggl(1 - \varepsilon^3\frac{8a_1}{3\pi}\biggr)}{1 + \varepsilon\frac{4a_0}{3\pi} + \varepsilon^2\Bigl(\frac{4a_0}{3\pi}\Bigr)^2 + \varepsilon^3\Bigl(\frac{4a_0}{3\pi}\Bigr)^3 + \varepsilon^4\Bigl(\frac{4a_0}{3\pi}\Bigr)^4 + \varepsilon^3\frac{8a_1}{15\pi}\biggl(1 + \varepsilon\frac{8a_0}{3\pi}\biggr)}. \tag{B.5}$$

The fourth-order asymptotic solution derived by Argatov and Sabina (2013), which is *asymptotically equivalent* to (B.2), (B.3), has the form

$$\tilde{F} = \frac{4}{3}\varepsilon^3\biggl(1 - \varepsilon^3\frac{8a_1}{3\pi}\biggr), \tag{B.6}$$

$$\tilde{\delta} = \varepsilon^2\biggl(1 - \varepsilon\frac{4a_0}{3\pi} - \varepsilon^3\frac{16a_1}{5\pi} + \varepsilon^4\frac{32a_0a_1}{9\pi^2}\biggr). \tag{B.7}$$

The second-order asymptotic solution is recovered from Eqs. (B.6) and (B.7) by dropping the terms containing $a_1$, that is as follows (Argatov, 2010):

$$\tilde{F} = \frac{4}{3}\varepsilon^3, \quad \tilde{\delta} = \varepsilon^2\biggl(1 - \varepsilon\frac{4a_0}{3\pi}\biggr). \tag{B.8}$$

Observe that Eqs. (B.2)–(B.7) utilze only the first two asymptotic constants $a_0$ and $a_1$. The sixth-order asymptotic solution derived by Argatov (2001), which incorporates also the third asymptotic constant $a_2$, has the following form:

$$\tilde{F} = \frac{4}{3}\varepsilon^3\biggl(1 - \varepsilon^3\frac{8a_1}{3\pi} - \varepsilon^5\frac{128a_2}{15\pi}\biggr), \tag{B.9}$$

$$\tilde{\delta} = \frac{\varepsilon^2\biggl(1 - \varepsilon^3\frac{8a_1}{3\pi} - \varepsilon^5\frac{128a_2}{15\pi}\biggr)}{\biggl(1 - \varepsilon\frac{4a_0}{3\pi}\biggr)^{-1} + \varepsilon^3\frac{8a_1}{15\pi}\biggl(1 + \varepsilon\frac{8a_0}{3\pi}\biggr) + \varepsilon^5\frac{128}{45\pi}\biggl(\frac{a_0a_1}{\pi} - \frac{a_2}{7}\biggr)}. \tag{B.10}$$

The fourth-order asymptotic approximation for the force-displacement relation in the *explicit* form, which was obtained by Argatov (2011), has the form

$$\tilde{F} = \frac{4}{3}\varpi^3\biggl\{1 + \varpi\frac{2a_0}{\pi} + \varpi^2\frac{14a_0^2}{3\pi^2} + \varpi^3\Bigl(\frac{320a_0^3}{27\pi^3} + \frac{32a_1}{15\pi}\Bigr) + \varpi^4\Bigl(\frac{286a_0^4}{9\pi^4} + \frac{64a_0a_1}{5\pi^2}\Bigr)\biggr\} \tag{B.11}$$

and simplifies to the second-order approximation as follows Argatov (2011):

$$\tilde{F} = \frac{4}{3}\varpi^3\biggl(1 + \varpi\frac{2a_0}{\pi} + \varpi^2\frac{14a_0^2}{3\pi^2}\biggr). \tag{B.12}$$

We recall that for a bonded incompressible layer, we have $a_0 = 1.77022$, $a_1 = -0.95777$, and $a_2 = 0.43736$. In this special case, the following approximate solution was obtained by Dimitriadis et al. (2002):

$$\tilde{F} = \frac{4}{3}\tilde{\delta}^{3/2}\left(1 + 1.133\tilde{\delta}^{1/2} + 1.283\tilde{\delta} + 0.769\tilde{\delta}^{3/2} + 0.0975\tilde{\delta}^2\right). \tag{B.13}$$

A much more accurate solution was derived by Garcia and Garcia (2018) in the form

$$\tilde{F} = \frac{4}{3}\tilde{\delta}^{3/2}\left(1 + 1.133\tilde{\delta}^{1/2} + 1.497\tilde{\delta} + 1.469\tilde{\delta}^{3/2} + 0.755\tilde{\delta}^2\right), \tag{B.14}$$

which differs from (B.13) only by the expansion coefficients.

The accuracy of the analytical solutions outlined above have been tested using the following accurate analytical approximation obtained by Hermanowicz (2021):

$$\tilde{F} = \begin{cases} \dfrac{4}{3}\tilde{\delta}^{3/2}\left(1 + 1.105\tilde{\delta}^{1/2} + 1.607\tilde{\delta} + 1.602\tilde{\delta}^{3/2}\right), & \tilde{\delta} \leq 0.4 \\ 0.616 - 3.114\tilde{\delta}^{1/2} + 6.693\tilde{\delta} - 7.17\tilde{\delta}^{3/2} + 8.228\tilde{\delta}^2 + \dfrac{\pi}{2}\tilde{\delta}^3, & 0.4 < \tilde{\delta}. \end{cases} \tag{B.15}$$

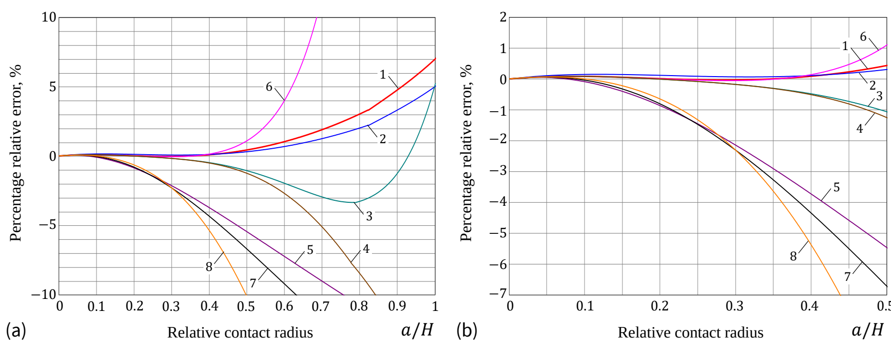


Figure 2: Accuracy of the approximate solutions as a function of the relative contact radius.

The results of the comparison are presented in Figs. 2 and 3, where the following legend applies: Curve 1 corresponds to the fourth-order asymptotic approximation in the explicit form (B.11) (Argatov, 2011); Curve 2 corresponds to the analytical approximation (B.14) (Garcia and Garcia, 2018); Curve 3 corresponds to the fourth-order asymptotic approximation in the parametric form (B.6), (B.6) (Argatov and Sabina, 2013); Curve 4 corresponds to the fourth-order asymptotic approximation in the parametric form (B.2), (B.3) (Vorovich et al., 1974); Curve 5 corresponds to the analytical approximation (B.13) (Dimitriadis et al., 2002); Curve 6 corresponds to the sixth-order asymptotic approximation in the parametric form (B.9), (B.10) (Argatov, 2001); Curve 7

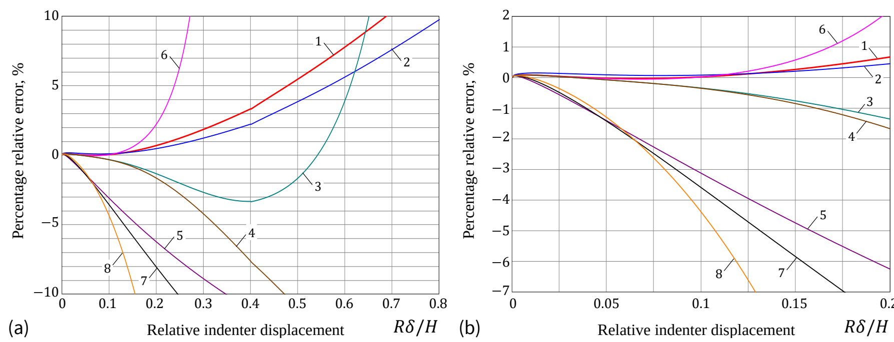


Figure 3: Accuracy of the approximate solutions as a function of the relative indentation depth.

corresponds to the second-order asymptotic approximation in the explicit form (B.12) (Argatov, 2011); Curve 8 corresponds to the second-order asymptotic approximation in the parametric form (B.8) (Argatov, 2010).

# C. Indentation scaling factor

According to the fourth-order asymptotic solution (B.11) obtained by Argatov (2011), the indentation scaling factor can be evaluated as

$$f(\varpi) = 1 + \varpi\frac{2a_0}{\pi} + \varpi^2\frac{14a_0^2}{3\pi^2} + \varpi^3\Big(\frac{320a_0^3}{27\pi^3} + \frac{32a_1}{15\pi}\Big) + \varpi^4\Big(\frac{286a_0^4}{9\pi^4} + \frac{64a_0a_1}{5\pi^2}\Big). \qquad \text{(B.16)}$$

In view of (A.1) and (A.2), the variation of the scaling factor $f$ as a function of $\varpi$ depends on the layer Poisson's ratio $\nu$. This is illustrated in Fig. 4.

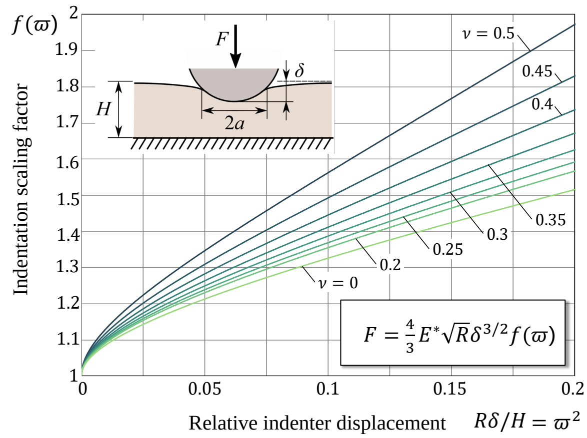


Figure 4: Indentation scaling factor for a paraboloidal indentation of a bonded elastic layer.

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