Supplementary Material for A note about convected time derivatives for flows of complex fluids

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S.1. INVARIANCE ASSOCIATED WITH CONVECTED TIME DERIVATIVES OF THE CONFORMATION TENSOR

Recall Sec. II in the manuscript on the kinematics of line elements, where the (Eulerian) time derivative $\overset{\triangledown}{\boldsymbol{\ell}}=\mathbf{0}$ following the flow ($\overset{\triangledown}{\boldsymbol{\ell}}=\frac{D\boldsymbol{\ell}}{Dt}-\boldsymbol{\ell}\cdot\boldsymbol{\nabla}\mathbf{u}$) implies the invariant solution $\boldsymbol{\ell}_L\cdot\mathbf{F}^{-1}=$ constant, from which the time variation of $\boldsymbol{\ell}_L(t;\mathbf{X})$ can be determined for a given flow or $\mathbf{F}$ (see Eq. (8)). Following similar steps, one can take a Lagrangian view and integrate the equation $\overset{\triangledown}{\mathbf{A}}=\frac{D\mathbf{A}}{Dt}-(\boldsymbol{\nabla}\mathbf{u})^T\cdot\mathbf{A}-\mathbf{A}\cdot\boldsymbol{\nabla}\mathbf{u}=\mathbf{0}$. We use the identities $\mathbf{F}^{-1}\cdot\mathbf{F}=\mathbf{I}$, $\mathbf{F}^T\cdot\mathbf{F}^{-T}=\mathbf{I}$, $\frac{\partial\mathbf{F}^{-T}}{\partial t}=-\mathbf{F}^{-T}\cdot(\boldsymbol{\nabla}\mathbf{u})^T$, $\frac{\partial\mathbf{F}}{\partial t}=\mathbf{F}\cdot\boldsymbol{\nabla}\mathbf{u}$, and take a Lagrangian notation with $\mathbf{A}(\mathbf{x},t)=\mathbf{A}_L(t;\mathbf{X})$ to write

$$\begin{aligned}\overset{\triangledown}{\mathbf{A}}&=\frac{\partial\mathbf{A}_L}{\partial t}-\mathbf{F}^T\cdot\mathbf{F}^{-T}\cdot(\boldsymbol{\nabla}\mathbf{u})^T\cdot\mathbf{A}_L-\mathbf{A}_L\cdot\mathbf{F}^{-1}\cdot\mathbf{F}\cdot\boldsymbol{\nabla}\mathbf{u}\\&=\frac{\partial\mathbf{A}_L}{\partial t}+\mathbf{F}^T\cdot\frac{\partial\mathbf{F}^{-T}}{\partial t}\cdot\mathbf{A}_L-\mathbf{A}_L\cdot\mathbf{F}^{-1}\cdot\frac{\partial\mathbf{F}}{\partial t}=\frac{\partial\mathbf{A}_L}{\partial t}+\mathbf{F}^T\cdot\frac{\partial\mathbf{F}^{-T}}{\partial t}\cdot\mathbf{A}_L+\mathbf{A}_L\cdot\frac{\partial\mathbf{F}^{-1}}{\partial t}\cdot\mathbf{F}.\end{aligned}\tag{S1}$$

Right multiplying by $\mathbf{F}^{-1}$ and left multiplying by $\mathbf{F}^{-T}$ yields

$$\mathbf{F}^{-T}\cdot\overset{\triangledown}{\mathbf{A}}\cdot\mathbf{F}^{-1}=\mathbf{F}^{-T}\cdot\frac{\partial\mathbf{A}_L}{\partial t}\cdot\mathbf{F}^{-1}+\frac{\partial\mathbf{F}^{-T}}{\partial t}\cdot\mathbf{A}_L\cdot\mathbf{F}^{-1}+\mathbf{F}^{-T}\cdot\mathbf{A}_L\cdot\frac{\partial\mathbf{F}^{-1}}{\partial t}=\frac{\partial}{\partial t}\left(\mathbf{F}^{-T}\cdot\mathbf{A}_L\cdot\mathbf{F}^{-1}\right).\tag{S2}$$

Thus, we conclude that $\overset{\triangledown}{\mathbf{A}}=\mathbf{0}$ implies

$$\frac{\partial}{\partial t}\left(\mathbf{F}^{-T}\cdot\mathbf{A}_L\cdot\mathbf{F}^{-1}\right)=\mathbf{0}\quad\text{or}\quad\mathbf{F}^{-T}\cdot\mathbf{A}_L\cdot\mathbf{F}^{-1}=\text{constant},\tag{S3}$$

following the fluid motion, i.e., it is an invariant. Using the initial data, $\mathbf{F}^{-T}\cdot\mathbf{A}_L\cdot\mathbf{F}^{-1}=\mathbf{A}_L(0;\mathbf{X})$ since $\mathbf{F}(0)=\mathbf{I}$, which means $\mathbf{A}_L=\mathbf{F}^T\cdot\mathbf{A}_L(0;\mathbf{X})\cdot\mathbf{F}$. For example, if $\mathbf{A}_L(0;\mathbf{X})=\mathbf{I}$, i.e., an undeformed isotropic state, then $\mathbf{A}_L=\mathbf{F}^T\cdot\mathbf{F}$.

Similarly, one can show for the lower-convected time derivative that $\overset{\vartriangle}{\mathbf{A}}=\mathbf{0}$ implies $\frac{\partial}{\partial t}\left(\mathbf{F}\cdot\mathbf{A}_L\cdot\mathbf{F}^T\right)=\mathbf{0}$. Thus, $\mathbf{F}\cdot\mathbf{A}_L\cdot\mathbf{F}^T$ is an invariant following the motion.

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S.2. INVARIANCE ASSOCIATED WITH THE CONSTRAINED CONVECTED TIME DERIVATIVE OF THE CONFORMATION TENSOR

It is possible to extend the ideas introduced above for identifying invariances following the fluid motion when $\overset{\triangledown}{\mathbf{A}}=\mathbf{0}$ to the case of $\overset{\triangledown}{\mathbf{A}}=-2\mathbf{A}\cdot\mathbf{E}\cdot\mathbf{A}$, corresponding to the constrained upper-convected time derivative.

Using a Lagrangian notation with $\mathbf{A}(\mathbf{x},t)=\mathbf{A}_L(t;\mathbf{X})$ and Eq. (S1), $\overset{\triangledown}{\mathbf{A}}=-2\mathbf{A}\cdot\mathbf{E}\cdot\mathbf{A}$ can be expressed as

$$\overset{\triangledown}{\mathbf{A}}=\frac{\partial \mathbf{A}_L}{\partial t}+\mathbf{F}^T\cdot\frac{\partial \mathbf{F}^{-T}}{\partial t}\cdot\mathbf{A}_L+\mathbf{A}_L\cdot\frac{\partial \mathbf{F}^{-1}}{\partial t}\cdot\mathbf{F}=-2\mathbf{A}_L\cdot\mathbf{E}\cdot\mathbf{A}_L. \tag{S4}$$

Right multiplying by $\mathbf{F}^{-1}$ and left multiplying by $\mathbf{F}^{-T}$ and using Eq. (S2), $\mathbf{F}^{-1}\cdot\mathbf{F}=\mathbf{I}$, and $\mathbf{F}^T\cdot\mathbf{F}^{-T}=\mathbf{I}$, Eq. (S4) yields

$$\frac{\partial}{\partial t}\left(\mathbf{F}^{-T}\cdot\mathbf{A}_L\cdot\mathbf{F}^{-1}\right)=-2\left(\mathbf{F}^{-T}\cdot\mathbf{A}_L\cdot\mathbf{F}^{-1}\right)\cdot\left(\mathbf{F}\cdot\mathbf{E}\cdot\mathbf{F}^T\right)\cdot\left(\mathbf{F}^{-T}\cdot\mathbf{A}_L\cdot\mathbf{F}^{-1}\right). \tag{S5}$$

Using the identities $\frac{\partial \mathbf{F}}{\partial t}=\mathbf{F}\cdot\nabla\mathbf{u}$ and $\frac{\partial \mathbf{F}^T}{\partial t}=(\nabla\mathbf{u})^T\cdot\mathbf{F}^T$, the term $2(\mathbf{F}\cdot\mathbf{E}\cdot\mathbf{F}^T)$ in Eq. (S5) can be expressed as

$$2\left(\mathbf{F}\cdot\mathbf{E}\cdot\mathbf{F}^T\right)=\mathbf{F}\cdot\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^T\right)\cdot\mathbf{F}^T=\frac{\partial \mathbf{F}}{\partial t}\cdot\mathbf{F}^T+\mathbf{F}\cdot\frac{\partial \mathbf{F}^T}{\partial t}=\frac{\partial}{\partial t}\left(\mathbf{F}\cdot\mathbf{F}^T\right), \tag{S6}$$

so that Eq. (S5) takes the form

$$\frac{\partial}{\partial t}\left(\mathbf{F}^{-T}\cdot\mathbf{A}_L\cdot\mathbf{F}^{-1}\right)=-\left(\mathbf{F}^{-T}\cdot\mathbf{A}_L\cdot\mathbf{F}^{-1}\right)\cdot\frac{\partial}{\partial t}\left(\mathbf{F}\cdot\mathbf{F}^T\right)\cdot\left(\mathbf{F}^{-T}\cdot\mathbf{A}_L\cdot\mathbf{F}^{-1}\right). \tag{S7}$$

or

$$\frac{\partial \mathbf{G}_L}{\partial t}=-\mathbf{G}_L\cdot\frac{\partial}{\partial t}\left(\mathbf{F}\cdot\mathbf{F}^T\right)\cdot\mathbf{G}_L\qquad\text{with}\qquad\mathbf{G}_L=\mathbf{F}^{-T}\cdot\mathbf{A}_L\cdot\mathbf{F}^{-1}. \tag{S8}$$

Under the assumption that $\mathbf{G}_L$ (in fact, $\mathbf{A}_L$) is *invertible* and using the identities $\mathbf{G}_L^{-1}=\mathbf{F}\cdot\mathbf{A}_L^{-1}\cdot\mathbf{F}^T$ and $\frac{\partial \mathbf{G}_L}{\partial t}=-\mathbf{G}_L\cdot\frac{\partial \mathbf{G}_L^{-1}}{\partial t}\cdot\mathbf{G}_L$, from Eq. (S8) it follows that

$$\frac{\partial \mathbf{G}_L^{-1}}{\partial t}=\frac{\partial}{\partial t}\left(\mathbf{F}\cdot\mathbf{F}^T\right)\qquad\text{or}\qquad\frac{\partial}{\partial t}\left(\mathbf{G}_L^{-1}-\mathbf{F}\cdot\mathbf{F}^T\right)=\frac{\partial}{\partial t}\left(\mathbf{F}\cdot\mathbf{A}_L^{-1}\cdot\mathbf{F}^T-\mathbf{F}\cdot\mathbf{F}^T\right)=\mathbf{0}. \tag{S9}$$

It follows that $\mathbf{F}\cdot\mathbf{A}_L^{-1}\cdot\mathbf{F}^T-\mathbf{F}\cdot\mathbf{F}^T=$ constant, following the fluid motion, i.e., it is an invariant. Using the initial data, we obtain $\mathbf{F}\cdot\mathbf{A}_L^{-1}\cdot\mathbf{F}^T-\mathbf{F}\cdot\mathbf{F}^T=\mathbf{A}_L^{-1}(0;\mathbf{X})-\mathbf{I}$ since $\mathbf{F}(0)=\mathbf{I}$, which means $\mathbf{A}_L^{-1}(t;\mathbf{X})=\mathbf{I}-\mathbf{F}^{-1}\cdot\mathbf{F}^{-T}+\mathbf{F}^{-1}\cdot\mathbf{A}_L^{-1}(0;\mathbf{X})\cdot\mathbf{F}^{-T}$.

Similarly, one can show for the constrained lower-convected time derivative, $\overset{\vartriangle}{\mathbf{A}}=2\mathbf{A}\cdot\mathbf{E}\cdot\mathbf{A}$, that

$$\frac{\partial}{\partial t}\left(\mathbf{F}\cdot\mathbf{A}_L\cdot\mathbf{F}^T\right)=2\left(\mathbf{F}\cdot\mathbf{A}_L\cdot\mathbf{F}^T\right)\cdot\left(\mathbf{F}^{-T}\cdot\mathbf{E}\cdot\mathbf{F}^{-1}\right)\cdot\left(\mathbf{F}\cdot\mathbf{A}_L\cdot\mathbf{F}^T\right). \tag{S10}$$

Using the identities $\frac{\partial \mathbf{F}^{-T}}{\partial t}=-\mathbf{F}^{-T}\cdot(\nabla\mathbf{u})^T$ and $\frac{\partial \mathbf{F}^{-1}}{\partial t}=-\nabla\mathbf{u}\cdot\mathbf{F}^{-1}$, the term $2(\mathbf{F}^{-T}\cdot\mathbf{E}\cdot\mathbf{F}^{-1})$ in Eq. (S10) can be expressed as

$$2\left(\mathbf{F}^{-T}\cdot\mathbf{E}\cdot\mathbf{F}^{-1}\right)=\mathbf{F}^{-T}\cdot\left(\nabla\mathbf{u}+(\nabla\mathbf{u})^T\right)\cdot\mathbf{F}^{-1}=-\mathbf{F}^{-T}\cdot\frac{\partial \mathbf{F}^{-1}}{\partial t}-\frac{\partial \mathbf{F}^{-T}}{\partial t}\cdot\mathbf{F}^{-1}=-\frac{\partial}{\partial t}\left(\mathbf{F}^{-T}\cdot\mathbf{F}^{-1}\right), \tag{S11}$$

so that Eq. (S10) takes the form

$$\frac{\partial}{\partial t}\left(\mathbf{F}\cdot\mathbf{A}_L\cdot\mathbf{F}^T\right)=-\left(\mathbf{F}\cdot\mathbf{A}_L\cdot\mathbf{F}^T\right)\cdot\frac{\partial}{\partial t}\left(\mathbf{F}^{-T}\cdot\mathbf{F}^{-1}\right)\cdot\left(\mathbf{F}\cdot\mathbf{A}_L\cdot\mathbf{F}^T\right), \tag{S12}$$

or

$$\frac{\partial \mathbf{H}_L}{\partial t}=-\mathbf{H}_L\cdot\frac{\partial}{\partial t}\left(\mathbf{F}^{-T}\cdot\mathbf{F}^{-1}\right)\cdot\mathbf{H}_L\qquad\text{with}\qquad\mathbf{H}_L=\mathbf{F}\cdot\mathbf{A}_L\cdot\mathbf{F}^T. \tag{S13}$$

Again, under the assumption that $\mathbf{H}_L$ (in fact, $\mathbf{A}_L$) is *invertible* and using the identities $\mathbf{H}_L^{-1}=\mathbf{F}^{-T}\cdot\mathbf{A}_L^{-1}\cdot\mathbf{F}^{-1}$ and $\frac{\partial \mathbf{H}_L}{\partial t}=-\mathbf{H}_L\cdot\frac{\partial \mathbf{H}_L^{-1}}{\partial t}\cdot\mathbf{H}_L$, from Eq. (S13) it follows that

$$\frac{\partial \mathbf{H}_L^{-1}}{\partial t}=\frac{\partial}{\partial t}\left(\mathbf{F}^{-T}\cdot\mathbf{F}^{-1}\right)\qquad\text{or}\qquad\frac{\partial}{\partial t}\left(\mathbf{H}_L^{-1}-\mathbf{F}^{-T}\cdot\mathbf{F}^{-1}\right)=\frac{\partial}{\partial t}\left(\mathbf{F}^{-T}\cdot\mathbf{A}_L^{-1}\cdot\mathbf{F}^{-1}-\mathbf{F}^{-T}\cdot\mathbf{F}^{-1}\right)=\mathbf{0}. \tag{S14}$$

Thus, $\mathbf{F}^{-T}\cdot\mathbf{A}_L^{-1}\cdot\mathbf{F}^{-1}-\mathbf{F}^{-T}\cdot\mathbf{F}^{-1}$ is an invariant following the motion.

S.3. LAGRANGIAN INTEGRATION OF THE OLDROYD-B CONSTITUTIVE EQUATION

The ideas introduced in Sec. II and III C for identifying invariances following the fluid motion can also be applied to the evolution equation for the conformation tensor when elastic stresses are included. For example, consider the Oldroyd-B constitutive equation (Eq. (36)), $\overset{\triangledown}{\mathbf{A}} = -\lambda_0^{-1}(\mathbf{A}-\mathbf{I})$, where λ_0 is a constant relaxation time. In this case, using Eq. (S1) and a Lagrangian notation with $\mathbf{A}(\mathbf{x},t) = \mathbf{A}_L(t;\mathbf{X})$, we have

$$\frac{\partial \mathbf{A}_L}{\partial t} + \mathbf{F}^T \cdot \frac{\partial \mathbf{F}^{-T}}{\partial t} \cdot \mathbf{A}_L + \mathbf{A}_L \cdot \frac{\partial \mathbf{F}^{-1}}{\partial t} \cdot \mathbf{F} = -\frac{1}{\lambda_0}(\mathbf{A}_L - \mathbf{I}). \tag{S15}$$

Now, recalling the steps just above, right multiplying by $\mathbf{F}^{-1}$ and left multiplying by $\mathbf{F}^{-T}$ yields

$$\frac{\partial}{\partial t}\left(\mathbf{F}^{-T} \cdot \mathbf{A}_L \cdot \mathbf{F}^{-1}\right) = -\frac{1}{\lambda_0}\mathbf{F}^{-T} \cdot (\mathbf{A}_L - \mathbf{I}) \cdot \mathbf{F}^{-1}. \tag{S16}$$

Thus, we find an ordinary differential equation for $\mathbf{F}^{-T} \cdot \mathbf{A}_L \cdot \mathbf{F}^{-1}$ with a time-dependent forcing,

$$\frac{\partial}{\partial t}\left(\mathbf{F}^{-T} \cdot \mathbf{A}_L \cdot \mathbf{F}^{-1}\right) + \frac{1}{\lambda_0}\mathbf{F}^{-T} \cdot \mathbf{A}_L \cdot \mathbf{F}^{-1} = \frac{1}{\lambda_0}\mathbf{F}^{-T} \cdot \mathbf{F}^{-1}. \tag{S17}$$

This equation is to be solved with initial data (suppressing dependence on $\mathbf{X}$) $\mathbf{F}^{-T} \cdot \mathbf{A}_L \cdot \mathbf{F}^{-1} = \mathbf{A}_L(0)$ since $\mathbf{F}(0) = \mathbf{I}$. Therefore, integrating, we obtain (see, e.g. [1, 2]),

$$\mathbf{F}^{-T}(t) \cdot \mathbf{A}_L(t) \cdot \mathbf{F}^{-1}(t) = \mathrm{e}^{-t/\lambda_0}\mathbf{A}_L(0) + \frac{1}{\lambda_0}\int_0^t \mathbf{F}^{-T}(t') \cdot \mathbf{F}^{-1}(t')\, \mathrm{e}^{-(t-t')/\lambda_0}\, \mathrm{d}t', \tag{S18}$$

or

$$\mathbf{A}_L(t) = \mathrm{e}^{-t/\lambda_0}\mathbf{F}^T(t) \cdot \mathbf{A}_L(0) \cdot \mathbf{F}(t) + \frac{1}{\lambda_0}\mathbf{F}^T(t) \cdot \int_0^t \mathbf{F}^{-T}(t') \cdot \mathbf{F}^{-1}(t')\, \mathrm{e}^{-(t-t')/\lambda_0}\, \mathrm{d}t' \cdot \mathbf{F}(t). \tag{S19}$$

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[2] J. H. Snoeijer, A. Pandey, M. A. Herrada, and J. Eggers, The relationship between viscoelasticity and elasticity, Proc. R. Soc. Lond. A **476**, 20200419 (2020).