Electronic Supplementary Material

July 21, 2023

1 Movie Caption

Active turbulence with deformable particles. Activity parameter is set to $\zeta = 0.0007$. (a) Aspect ratio, ω of particles. Colour scale indicates its magnitude. (b) Nematic director field $\mathbf{n}$ and scalar order parameter S. Colour scale indicates the magnitude of S. Red/blue symbols correspond to the core of $\pm 1/2$ defects respectively. $+1/2$ defects are oriented such that the tail is along the line shown. (c) Velocity field, $\mathbf{u}$. Colour scale indicates the magnitude of the velocity. (d) Projection of the strain rate along the director field, $E_{\parallel} = n_i E_{ij} n_j$. Colour scale indicates its sign and magnitude.

2 Dispersion Relation Derivation

The inverse Fourier transform for a fluctuating field f' is given by

$$f'(\mathbf{r}, t) = \int d\lambda dq \tilde{f}(\mathbf{q}, \lambda) e^{i\mathbf{q}\cdot\mathbf{r}+\lambda t}. \tag{1}$$

The time evolution equation for the tensorial order parameter $\mathbf{Q}$ is eliminated by setting $\mathbf{n} = \hat{x}$ and $S = (\omega - 1)/\omega$. The relevant equations of motion to first order in the perturbed fields are then:

$$\partial_t \omega' = 2E'_{\parallel} - \Gamma_\omega A_\omega \omega' + \frac{3}{2}\Gamma_\omega K_{LC} \nabla^2 \omega', \tag{2}$$

$$\rho \partial_t u'_i = \partial_j \left(2\eta E'_{ij} - p'\delta_{ij} - \frac{3}{2}\zeta\omega'(n_i n_j - \frac{1}{3}\delta_{ij}) \right). \tag{3}$$

In Fourier space, incompressibility reads

$$q_j \tilde{u}_j = 0 \Rightarrow \tilde{u}_x = -\frac{\sin\theta}{\cos\theta}\tilde{u}_y. \tag{4}$$

In Fourier space, Eq. (3) reads:

$$\rho\lambda\tilde{u}_i = \eta \left(-q_i \underbrace{q_j \tilde{u}_j}_{=0} - q^2 \tilde{u}_i \right) - q_i \tilde{p} - \frac{3i}{2}\zeta\tilde{\omega} q_j \left(n_i n_j - \frac{1}{3}\delta_{ij} \right). \tag{5}$$

Taking $\mathbf{n} = (1, 0, 0)$ and $\mathbf{q} = q(\cos\theta, \sin\theta, 0)$ where θ is the angle between $\mathbf{n}$ and $\mathbf{q}$ and considering $i = x$, we get

$$\rho\lambda\tilde{u}_x = -\eta q^2 \tilde{u}_x - iq\cos\theta\tilde{p} - i\zeta\tilde{\omega} q\cos\theta, \tag{6}$$

from which we can solve for $\tilde{p}$

$$\tilde{p} = \frac{i}{q\cos\theta}\left(\rho\lambda + \eta q^2\right)\tilde{u}_x - \zeta\tilde{\omega} = -\zeta\tilde{\omega} - \frac{i\sin\theta}{q\cos^2\theta}(\rho\lambda + \eta q^2)\tilde{u}_y, \tag{7}$$

where we have used incompressibility (4) in the final step. Now consider $i = y$

$$\rho\lambda\tilde{u}_y = -\eta q^2\tilde{u}_y - iq\sin\theta\tilde{p} - \frac{3i}{2}\zeta\tilde{\omega}\left(-\frac{1}{3}q\sin\theta\right). \tag{8}$$

Substituting the expression (7) for $\tilde{p}$

$$\left(\rho\lambda + \eta q^2\right)\tilde{u}_y = -\frac{\sin^2\theta}{\cos^2\theta}\left(\rho\lambda + \eta q^2\right)\tilde{u}_y + iq\sin\theta\zeta\tilde{\omega} + \frac{i}{2}q\sin\theta\zeta\tilde{\omega}, \tag{9}$$

allows us to solve for $\tilde{u}_y$

$$\tilde{u}_y = \frac{3iq\cos^2\theta\sin\theta\zeta\tilde{\omega}}{2\left(\rho\lambda + \eta q^2\right)}. \tag{10}$$

Next, we consider Eq. (2) in Fourier space:

$$\lambda\tilde{\omega} = 2iq\cos\theta\tilde{u}_x - \Gamma_\omega A_\omega\tilde{\omega} - \frac{3}{2}\Gamma_\omega K_{LC}q^2\tilde{\omega}. \tag{11}$$

Substituting for $\tilde{u}_x$ by using (10) for $\tilde{u}_y$ and the incompressibility relation (4), we finally get

$$\lambda\tilde{\omega} = \frac{3q^2\cos^2\theta\sin^2\theta\zeta}{(\rho\lambda + \eta q^2)}\tilde{\omega} - \Gamma_\omega A_\omega\tilde{\omega} - \frac{3}{2}\Gamma_\omega K_{LC}q^2\tilde{\omega}, \tag{12}$$

which, after some algebraic manipulation and a double angle formula gives the dispersion relation

$$q^2\left(\left(\eta + \frac{3}{2}\Gamma_\omega K_{LC}\rho\right)\lambda + \Gamma_\omega A_\omega\eta - \frac{3}{4}\zeta\sin^2 2\theta\right) + \frac{3}{2}\Gamma_\omega K_{LC}\eta q^4 + \rho\lambda^2 + \Gamma_\omega A_\omega\rho\lambda = 0. \tag{13}$$

For a solution with $\lambda > 0$ to exist for some value of q, we require the coefficient of q^2 to be negative. There is then a critical value of the activity, below which an isotropic system will be stable, given by

$$\zeta_c = \frac{4\Gamma_\omega A_\omega\eta}{3\sin^2 2\theta}. \tag{14}$$

ζ_c is clearly minimised when $\theta = \pi/4$. An analytical expression may be obtained for the most unstable wave vector, q_m which corresponds to the local maximum of the growth rate, λ_m. This is given by:

$$q_m^2 = \frac{\sqrt{2\rho K_{LC}\Gamma_\omega\zeta\sin^2 2\theta J_+^2 R} - 4\rho K_{LC}\Gamma_\omega\eta T}{2K_{LC}\Gamma_\omega\eta J_-^2}, \tag{15}$$

$$J_\pm = 2\eta \pm 3\rho\Gamma_\omega K_{LC}, \tag{16}$$

$$R = 3\zeta\sin^2 2\theta - 4\Gamma_\omega A_\omega\eta + 6\rho A_\omega K_{LC}\Gamma_\omega^2, \tag{17}$$

$$T = 3\zeta\sin^2 2\theta - \Gamma_\omega A_\omega J_-. \tag{18}$$

Despite the complicated form of the above expression, the competition between activity versus elasticity and energy is shown to be a determining factor for the most unstable modes.
Finally, by setting $\omega' = \epsilon \sin \mathbf{q} \cdot \mathbf{r}$ and using the Fourier transform of the sin function, we may also find a solution of the flow field, $\tilde{u}_y$.

$$\tilde{u}_y = \frac{3iq\cos^2\theta \sin\theta\zeta}{2(\rho\lambda + \eta q^2)} \frac{1}{2i} (2\pi)^2 \epsilon \left(\delta(\mathbf{q} + \mathbf{q}') - \delta(\mathbf{q} - \mathbf{q}')\right). \tag{19}$$

Setting $\lambda = 0$ for simplicity, $\theta = \pi/4$, and inverting the Fourier transform, we may get the full flow field:

$$u_y = -\frac{3}{4\sqrt{2}} \frac{\zeta\epsilon}{\eta q} \cos\left(\frac{1}{\sqrt{2}}(x+y)\right), \tag{20}$$

$$u_x = \frac{3}{4\sqrt{2}} \frac{\zeta\epsilon}{\eta q} \cos\left(\frac{1}{\sqrt{2}}(x+y)\right). \tag{21}$$