Supplemental Material: Electrostatic force on a spherical particle confined between two planar surfaces

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CONTENTS

I. IMAGE OF AN ELECTROSTATIC MULTIPOLE NEAR A CONDUCTING OR AN INSULATING PLANE WALL

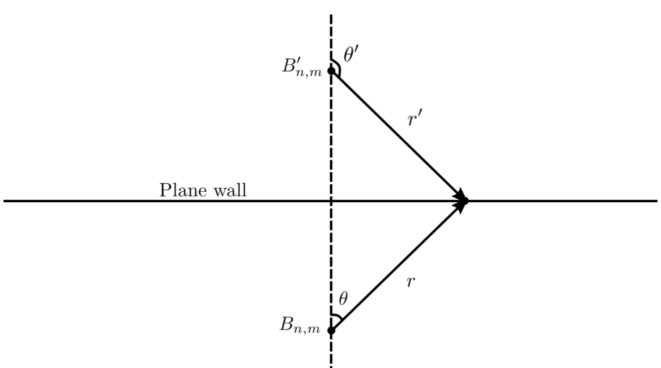

FIG. 1. Image

In this part, we give the relation between a multipole $B_{n,m}$ and its image $B'_{n,m}$ due to a grounded or insulating plane wall. These results follow Washizu and Jones [1] and are summarized here. The electric potential can be written in the following form

$$\Phi = \frac{B_{n,m}}{r^{n+1}} P_n^m(\cos\theta)\cos(m\phi) + \frac{B'_{n,m}}{r'^{n+1}} P_n^m(\cos\theta')\cos(m\phi'). \quad (1)$$

a. Grounded plane wall In the case of grounded wall, the boundary condition on the plane wall is $\Phi = 0$. $B_{n,m}$ and $B'_{n,m}$ are symmetric about the wall. Consequently, on the wall, we have $r = r'$, $\theta + \theta' = \pi$, and $\phi = \phi'$. Substituting into Eq. 1 yields

$$\frac{B_{n,m}}{r^{n+1}} P_n^m(\cos\theta)\cos(m\phi) + \frac{B'_{n,m}}{r^{n+1}} P_n^m(-\cos\theta)\cos(m\phi) = 0$$

Using the identity of associated Legendre polynomials $P_n^m(-x) = (-1)^{n+m} P_n^m(x)$, we have

$$B'_{n,m} = (-1)^{n+m+1} B_{n,m}. \quad (2)$$

b. Insulating plane wall In the case of insulating wall, the boundary condition on the plane wall is $\partial\Phi/\partial z = 0$. The partial derivative between with respect to z could be calculated using the following equation,

$$\frac{\partial}{\partial z}\left[\frac{P_n^m(\cos\theta)}{r^{n+1}}\cos(m\phi)\right] = -(n-m+1)\frac{P_{n+1}^m(\cos\theta)}{r^{n+2}}\cos(m\phi).$$

On the insulating wall, we have

$$B_{n,m}\left[-(n-m+1)\frac{P_{n+1}^m(\cos\theta)}{r^{n+2}}\cos(m\phi)\right] + B'_{n,m}\left[-(n-m+1)\frac{P_{n+1}^m(-\cos\theta)}{r^{j+2}}\cos(m\phi)\right] = 0.$$

Same as before, using the identity of associated Legendre polynomials $P_n^m(-x) = (-1)^{n+m} P_n^m(x)$, we have

$$B'_{n,m} = (-1)^{n+m} B_{n,m}. \quad (3)$$

II. RE-EXPANSION OF THE SPHERICAL HARMONICS

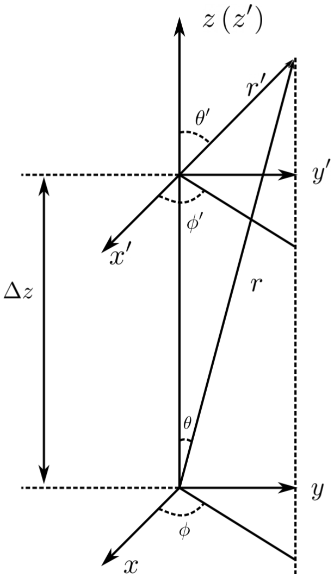

FIG. 2. Re-expansion

In this section, we present the re-expansion of spherical harmonics from spherical coordinates (r', θ', ϕ') to coordinates (r, θ, ϕ). These formulas are derived by Washizu and Jones [1]. Cartesian coordinates (x', y', z') and (x, y, z) are also introduced as auxiliary. z-axis of these two Cartesian coordinates are collinear and in the same direction. As shown in FIG. 2, Δz is the distance between origins of two coordinates.

We use $\psi_{n,m}$ and $\phi_{n,m}$ to note spherical harmonics with singularity at $r' = 0$ and $r = \infty$, respectively,

$$\psi_{l,m} = r'^{-l-1} P_l^m(\cos\theta') \cos(m\phi'),$$
$$\phi_{n,m} = r^n P_n^m(\cos\theta) \cos(m\phi).$$

For $\Delta z > 0$, which means that the coordinates (r', θ', ϕ') is above coordinates (r, θ, ϕ), the re-expansion is

$$\psi_{n,m} = \sum_{l=m}^{\infty} (-1)^{n-m} \frac{(n+l)!\,(\Delta z)^{-(n+l+1)}}{(n-m)!(l+m)!} \phi_{l,m}.$$

For $\Delta z < 0$, which means that the coordinates (r', θ', ϕ') is below coordinates (r, θ, ϕ) the re-expansion is

$$\psi_{n,m} = -\sum_{l=m}^{\infty} (-1)^{n-m} \frac{(n+l)!\,(\Delta z)^{-(n+l+1)}}{(n-m)!(l+m)!} \phi_{l,m}.$$

They could be rewritten in a more compact form

$$\psi_{n,m} = \sum_{l=m}^{\infty} (-1)^{n-m} \frac{(n+l)!}{(n-m)!(l+m)!} \frac{(\Delta z)^{-(n+l)}}{|\Delta z|} \phi_{l,m}. \tag{4}$$

It should be noted that Eq. 4 converges within the region $r < \Delta z$.

III. ELECTRIC POTENTIAL Φ_2

In this section, we give the expression of the disturbance electric potential outside the particle, $\hat{\Phi}_2$, using the superposition of multipole images. The electric potential could be written as

$$\hat{\Phi}_2 = \hat{\phi}_B + \sum_{\text{Images}} \hat{\phi}_I \tag{5}$$

The first term on the R.H.S. of Eq. 5 is the disturbance field due to the original multipole $B_{n,m}$,

$$\hat{\phi}_B = \sum_{n=m}^{\infty} \frac{B_{n,m}}{r^{n+1}} P_n^m(\cos\theta) \cos(m\phi),$$

where (r, θ, ϕ) is the spherical coordinates originated at the particle center. The second term on the R.H.S of Eq. 5 is the sum of the disturbance field due to all images. For an arbitrary image I, located at z_I with multipole moments $I_{n,m}$, the disturbance field $\hat{\phi}_I$ is

$$\hat{\phi}_I = \sum_{n=m}^{\infty} \frac{I_{n,m}}{r_I^{n+1}} P_n^m(\cos\theta_I) \cos(m\phi_I),$$

where (r_I, θ_I, ϕ_I) is the spherical coordinates originated at the particle center.

Using the re-expansion formulae in Eq. 4, $\hat{\phi}_I$ is re-expanded in coordinates (r, θ, ϕ) in the following form

$$\hat{\phi}_I = \sum_{n=m}^{\infty} \sum_{l=m}^{\infty} \bar{D}_{n,l}^{(m)} B_{l,m} r^n P_n^m(\cos\theta) \cos(m\phi),$$

where the re-expansion coefficients $\bar{D}_{n,l}^{(m)}$ are

$$\bar{D}_{n,l}^{(m)} = (-1)^{l-m} \frac{I_{l,m}}{B_{l,m}} \frac{(n+l)!}{(l-m)!(n+m)!} \frac{(\Delta z_I)^{-(n+l)}}{|\Delta z_I|}.$$

Substituting the re-expanded $\hat{\phi}_I$ into Eq. 5, we obtain

$$\hat{\Phi}_2 = \sum_{n=m}^{\infty} \frac{B_{n,m}}{r^{n+1}} P_n^m(\cos\theta) \cos(m\phi) + \sum_{\text{Images}} \sum_{n=m}^{\infty} \sum_{l=m}^{\infty} \bar{D}_{n,l}^{(m)} B_{l,m} r^n P_n^m(\cos\theta) \cos(m\phi).$$

The electric field outside the particle could also be written as the general solution of Laplace's equation,

$$\hat{\Phi}_2 = \sum_{n=m}^{\infty} \left(\frac{B_{n,m}}{r^{n+1}} + M_{n,m} r^n \right) P_n^m(\cos\theta) \cos(m\phi).$$

Equating two expressions gives

$$M_{n,m} = \sum_{l=m}^{\infty} N_{n,l}^{(m)} B_{l,m},$$

where coefficients $N_{n,l}^{(m)}$ are defined as

$$N_{n,l}^{(m)} = \sum_{\text{Images}} \bar{D}_{n,l}^{(m)}.$$

It should be mentioned that in the expressions of $\bar{D}_{n,l}^{(m)}$ the ratio $I_{l,m}/B_{l,m} = \pm 1$. The analysis on the image system in the main text gives the multipole moments and locations of all the images. We explicitly give the expression of $N_{n,l}^{(0)}$ for the normal electric filed and $N_{n,l}^{(1)}$ for the tangential electric field,

$$N_{n,l}^{(0)} = \frac{(n+l)!}{l!n!}\left[\frac{1}{(-2\delta_c)^{n+l+1}} + \sum_{k=1}^{\infty}\frac{(-1)^l + (-1)^n}{(2k\delta_w)^{n+l+1}} - \frac{1}{(2k\delta_w - 2\delta_c)^{n+l+1}} - \frac{(-1)^{n+l}}{(2k\delta_w + 2\delta_c)^{n+l+1}}\right], \tag{6}$$

$$N_{n,l}^{(1)} = -\frac{(n+l)!}{(l-1)!(n+1)!}\left[\frac{1}{(-2\delta_c)^{n+l+1}} + \sum_{k=1}^{\infty}\frac{(-1)^l + (-1)^n}{(2k\delta_w)^{n+l+1}} - \frac{1}{(2k\delta_w - 2\delta_c)^{n+l+1}} - \frac{(-1)^{n+l}}{(2k\delta_w + 2\delta_c)^{n+l+1}}\right]. \tag{7}$$

IV. ASYMPTOTIC SOLUTION FOR LARGE GAP BETWEEN TWO WALLS

The non-monotonic behavior of the force coefficient C_f presented in the main text is superising. In the main text, it is concluded that this behavior is the result of increasing dipole moment as the top wall approaches the particle. In this section, we present the asymptotic solution when $\delta_w \gg 1$. Without loss of generality, we assume the particle is close to the bottom wall, $\delta_c \sim \mathcal{O}(1)$.

From Eq. 6 and Eq. 7, it is clear that $N_{n,l}^{(m)}$ could be written as

$$N_{n,l}^{(m)} = X_{n,l}^{(m)} + Y_{n,l}^{(m)},$$

where $X_{n,l}^{(m)}$ stands for the leading order term that is independent of δ_w, and $Y_{n,l}^{(m)}$ is the correction term that comes from sums on the R.H.S of Eq. 6 and Eq. 7. We are only looking for the leading order correction,

$$Y_{n,l}^{(m)} = \frac{1}{\delta_w^{\eta}} Z_{n,l}^{(m)} + \text{h.o.t.}$$

It could be shown that $\eta = 3$ in both cases. The original multipole $B_{l,m}$ could also be expressed as a regular perturbation series,

$$B_{l,m} = (B_0)_{l,m} + \frac{1}{\delta_w^3}(B_3)_{l,m} + \text{h.o.t.}$$

$(B_0)_{l,m}$ and $(B_3)_{l,m}$ are solved by substituting $N_{n,l}^{(m)}$ and $B_{l,m}$ into the equation systems for the original multipole and equating like powers of $1/\delta_w$. The interaction force have the one-wall solution as the leading order term and a correction term of $\mathcal{O}(1/\delta_w^3)$,

$$C_f \sim C_{f0} + \frac{1}{\delta_w^3} C_{f3}.$$

Equations for the normal and tangential electric field are listed in the following two subsections respectively.

A. Normal electric field

It is mentioned that the net charge on the particle is zero and terms with subscript $n = 0$ or $l = 0$ do not contribute to the interaction force. From Eq. 6, it is obvious that

$$X_{n,l}^{(0)} = \frac{(n+l)!}{l!n!}(-2\delta_c)^{-(n+l+1)}.$$

The leading order correction term that contributes is $Y_{1,1}^{(0)}$,

$$Y_{1,1}^{(0)} = -\frac{1}{\delta_w^3}\sum_{k=1}^{\infty}\frac{1}{k^3} + \text{h.o.t} \sim -\frac{\zeta(3)}{\delta_w^3},$$

where $\zeta(x)$ is the Riemann zeta function. It could be verified that $Y_{1,1}^{(0)}$ is the only term of $\mathcal{O}(1/\delta_w^3)$. The asymptotic behavior of $N_{n,l}^{(0)}$ is

$$N_{n,l}^{(0)} \sim X_{n,l}^{(0)} + \frac{1}{\delta_w^3} Z_{n,l}^{(0)}, \tag{8}$$

where $Z_{n,l}^{(0)}$ is

$$Z_{n,l}^{(0)} = \begin{cases} -\zeta(3) & (n,l) = (1,1), \\ 0 & \text{otherwise.} \end{cases}$$

$(B_0)_{l,m}$ and $(B_3)_{l,m}$ are solved from the following equations

$$\mathcal{O}(1) : \sum_{l=1}^{\infty}\left[(\chi-1)nX_{n,l}^{(0)} + (n\chi+n+1)\delta_{n,l}\right](B_0)_{l,0} = (\chi-1)G_{n,0},$$

$$\mathcal{O}\left(\frac{1}{\delta_w^3}\right) : \sum_{l=1}^{\infty}\left[(\chi-1)nX_{n,l}^{(0)} + (n\chi+n+1)\delta_{n,l}\right](B_3)_{l,0} = -(\chi-1)n\sum_{l=1}^{\infty} Z_{n,l}^{(0)}(B_0)_{l,0}.$$

Substituting into the equation for the electric force in the main text, we obtain the asymptotic behavior of interaction force on the particle

$$\begin{aligned} C_f \sim & C_{f0} + \frac{1}{\delta_w^3} C_{f3}, \\ C_{f0} = & 4\pi \sum_{n=1}^{\infty} K_n \frac{n+1}{2n+1} \left(M_0'\right)_{n,0} \left(M_0'\right)_{n+1,0}, \\ C_{f3} = & 4\pi \sum_{n=1}^{\infty} K_n \frac{n+1}{2n+1} \Big[\left(M_0'\right)_{n,0} \left(M_3\right)_{n+1,0} \\ & + \left(M_3\right)_{n,0} \left(M_0'\right)_{n+1,0} \Big], \end{aligned}$$

where M coefficients are,

$$\begin{aligned} M_{n,0} &\sim \left(M_0\right)_{n,0} + \frac{1}{\delta_w^3} \left(M_3\right)_{n,0}, \\ \left(M_0\right)_{n,0} &= \sum_{l=1}^{\infty} X_{n,l}^{(0)} \left(B_0\right)_{l,0} \\ \left(M_3\right)_{n,0} &= \sum_{l=1}^{\infty} X_{n,l}^{(0)} \left(B_3\right)_{l,0} + \sum_{l=1}^{\infty} Z_{n,l}^{(0)} \left(B_0\right)_{l,0} \end{aligned}$$

B. Tangential electric field

In the case of tangential electric field, the asymptotic solution could be obtained following the same procedure. The leading order term $X_{n,l}^{(1)}$ is

$$X_{n,l}^{(1)} = -\frac{(n+l)!}{(l-1)!(n+1)!} (-2\delta_c)^{-(n+l+1)}.$$

The correction term is still of $\mathcal{O}(1/\delta_w^3)$,

$$N_{n,l}^{(1)} \sim X_{n,l}^{(1)} + \frac{1}{\delta_w^3} Z_{n,l}^{(1)},$$

where $Z_{n,l}^{(1)}$ is

$$Z_{n,l}^{(1)} = \begin{cases} \zeta(3)/2 & (n,l) = (1,1), \\ 0 & \text{otherwise}. \end{cases}$$

The leading order multipole $\left(B_0\right)_{l,1}$ and correction $\left(B_3\right)_{l,1}$ could be solved from following linear systems,

$$\begin{aligned} \mathcal{O}(1) : \sum_{l=1}^{\infty} \Big[(\chi-1) n X_{n,l}^{(1)} + (n\chi+n+1)\delta_{n,l} \Big] \left(B_0\right)_{l,1} \\ = (\chi-1) G_{n,1}, \\ \mathcal{O}\left(\frac{1}{\delta_w^3}\right) : \sum_{l=1}^{\infty} \Big[(\chi-1) n X_{n,l}^{(1)} + (n\chi+n+1)\delta_{n,l} \Big] \left(B_3\right)_{l,1} \\ = -(\chi-1) n \sum_{l=1}^{\infty} Z_{n,l}^{(1)} \left(B_0\right)_{l,1}. \end{aligned}$$

The asymptotic behavior of interaction force on the particle is

$$\begin{aligned} C_f \sim & C_{f0} + \frac{1}{\delta_w^3} C_{f3}, \\ C_{f0} = & 2\pi \sum_{n=1}^{\infty} K_n \frac{n(n+1)(n+2)}{2n+1} \left(M_0'\right)_{n,1} \left(M_0'\right)_{n+1,1}, \\ C_{f3} = & 2\pi \sum_{n=1}^{\infty} K_n \frac{n(n+1)(n+2)}{2n+1} \\ & \times \Big[\left(M_0'\right)_{n,1} \left(M_3\right)_{n+1,1} + \left(M_3\right)_{n,1} \left(M_0'\right)_{n+1,1} \Big], \end{aligned}$$

where M coefficients are,

$$\begin{aligned} M_{n,1} &\sim \left(M_0\right)_{n,1} + \frac{1}{\delta_w^3} \left(M_3\right)_{n,1}, \\ \left(M_0\right)_{n,1} &= \sum_{l=1}^{\infty} X_{n,l}^{(1)} \left(B_0\right)_{l,1}, \\ \left(M_3\right)_{n,1} &= \sum_{l=1}^{\infty} X_{n,l}^{(1)} \left(B_3\right)_{l,1} + \sum_{l=1}^{\infty} Z_{n,l}^{(1)} \left(B_0\right)_{l,1}. \end{aligned}$$

V. CONVERGENCE

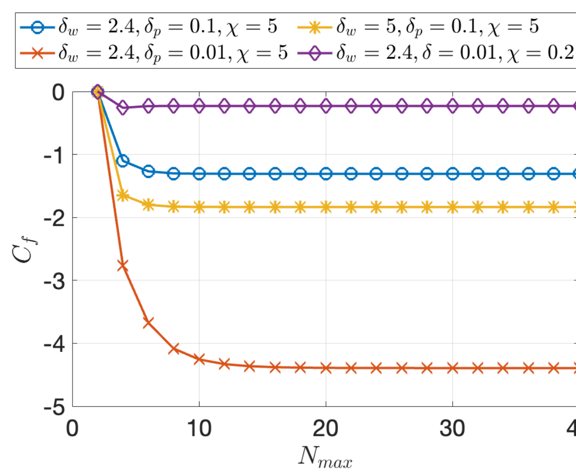

FIG. 3. The convergence of force coefficient C_f in the case of normal electric field for different configurations.

In this section, we present the convergence of our analytical solution in the normal electric field. In practical computation procedure, we truncate the infinit sum Eq.25 in the main text and keep finite number terms. The number is noted as N_{max}. In FIG. 3, we plot the force coefficient C_f as a function of N_{max}. It could seen from the figure that all the solutions converge when enough terms are kept.

[1] Masao Washizu and Thomas B Jones. Dielectrophoretic interaction of two spherical particles calculated by equivalent multipole-moment method. IEEE Transactions on Industry Applications, 32(2):233–242, 1996.