Curvature screening in draped mechanical metamaterial sheets
Supporting Information

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1 Boundary layer analysis

1.1 Dilational metamaterial

For dilational metamaterial sheets, the governing equations of motion in the Föppl-Von Kármán limit are

$$\triangle(\chi/Y) = (\kappa_0/Y)A - (\kappa_1/Y)\Delta A, \tag{1}$$
$$\triangle^2(\chi/Y) = -G - \Delta A \tag{2}$$

where κ_1/YL^2 is assumed to be a small, dimensionless number, where L is the dimension of the sheet. Here $Y = 4\mu$ as per the main text. And The boundary conditions are

$$\hat{\mathbf{n}} \cdot \nabla A|_B = 0, \tag{3}$$
$$\sigma \cdot \hat{\mathbf{n}}|_B = 0, \tag{4}$$

where σ is the stress tensor, $|_B$ denotes the restriction of the functions to the boundary, $\hat{\mathbf{n}}$ is the unit vector normal to the boundary but tangent to the metamaterial sheet, and repeated indices are summed.

The presence of a small parameter κ_1/YL^2 multiplying the highest derivative of A suggests the solution will take the form of a slowly varying bulk term and a rapidly varying boundary layer. An approximate solution may be obtained as described in the next sections.

1.1.1 Outside the boundary layer

Since $\varepsilon = \frac{\kappa_1}{YL^2}$ is small, we look for a solution in the bulk of the sheet of the form,

$$A = A_0 + \varepsilon A_1 + O(\varepsilon^2) \tag{5}$$
$$\chi = \chi_0 + \varepsilon\chi_1 + O(\varepsilon^2). \tag{6}$$

Substituting into Eqs. (1) and (2), we obtain

$$\triangle\chi_0 = \kappa_0 A_0 \tag{7}$$
$$\triangle A_0 = -\frac{Y}{Y+\kappa_0}G \tag{8}$$

to lowest order in ε. Assuming axisymmetry and setting $Y = 4\mu$ we therefore obtain $a_2 = -\frac{\mu G}{(4\mu+\kappa_0)}$, thus giving

$$A_0 = -\frac{\mu G}{4\mu+\kappa_0}r^2 + a_0, \tag{9}$$

,where a_0 is a constant.

1.1.2 Inside the boundary layer

To obtain the solution near the boundary, assume the boundary layer has thickness δ and make the coordinate substitution $\eta = (R-r)/\delta$. From the equations of motion, we obtain the following effective equation in A

$$\frac{\kappa_1/4\mu}{\delta^2}\triangle^2 A_{in} = (1+\frac{\kappa_0}{4\mu})\triangle A_{in}. \tag{10}$$

Setting the boundary layer thickness $\delta = \sqrt{\frac{\kappa_1}{4\mu+\kappa_0}} = l_{sc}$, we obtain

$$A(\eta)_{in} = c_1(I_0(\eta)-1) + c_2 Y_0(-i\eta) \tag{11}$$

1.1.3 Full solution

Imposing the boundary condition $A'(R) = 0$ on the boundary layer solution, in the limit $\delta \to 0$ that we can evaluate A

$$A = a_0 - \frac{4\mu G}{4\mu + \kappa_0} \left(\frac{r^2}{4} - l_{sc} \frac{R}{2} \frac{I_0\left(\frac{r}{l_{sc}}\right)}{I_1\left(\frac{R}{l_{sc}}\right)} \right) \tag{12}$$

The coefficient a_0 can be obtained by minimizing the energy for A we have obtained. Therefore, up to leading order in the screening length we have

$$A = \frac{\mu G(R^2 - 12 l_{sc}^2)}{3(\kappa_0 + 4\mu)} - \frac{4\mu G}{4\mu + \kappa_0} \left(\frac{r^2}{4} - l_{sc} \frac{R}{2} \frac{I_0\left(\frac{r}{l_{sc}}\right)}{I_1\left(\frac{R}{l_{sc}}\right)} \right) \tag{13}$$

Similarly to ensure vanishing normal and shear stresses on the boundary, i.e, $\chi'(R) = 0$, we get χ up to appropriate order in δ

$$\chi = d_0 + \frac{G\mu\kappa_0 R^2}{12(\kappa_0 + 4\mu)} r^2 - \frac{G\mu\kappa_0}{16(\kappa_0 + 4\mu)} r^4 + l_{sc}^2 \frac{\mu G}{4\mu + \kappa_0} \left(4\mu r^2 - \frac{R}{2} \frac{I_0\left(\frac{r}{l_{sc}}\right)}{I_1\left(\frac{R}{l_{sc}}\right)} \right) \tag{14}$$

Note that the harmonic function in χ in the main text with the choice of boundary conditions reduces to the constant d_0.

1.2 Simple shear metamaterial

For simple shear sheets, in Cartesian coordinates, the equilibrium equations are

$$\partial_1 \partial_2 \chi = -\kappa_0 A + \kappa_1 \triangle A \tag{15}$$

$$\frac{1}{Y} \triangle^2 \chi - \frac{1}{\mu} \partial_1^2 \partial_2^2 \chi = -G + \partial_1 \partial_2 A \tag{16}$$

1.2.1 Outside the boundary layer

Outside the boundary layer,

$$A = A_0 + \varepsilon A_1 + O(\varepsilon^2), \tag{17}$$

$$\chi = \chi_0 + \varepsilon \chi_1 + O(\varepsilon^2) \tag{18}$$

and so

$$\partial_1 \partial_2 \chi_0 = -\kappa_0 A_0, \tag{19}$$

$$\frac{1}{Y} \triangle^2 \chi_0 - \frac{1}{\mu} \partial_1^2 \partial_2^2 \chi_0 = -G + \partial_1 \partial_2 A_0. \tag{20}$$

We obtain the following working equation in χ_0,

$$\frac{1}{Y} \left(\partial_1^4 + \partial_2^4 \right) \chi_0 + \left(\frac{1}{\kappa_0} + 2\frac{1}{Y} - \frac{1}{\mu} \right) \partial_1^2 \partial_2^2 \chi_0 = -G. \tag{21}$$

We can guess the form of the simplest solution for the limit $\kappa_1 \to 0$ and $\kappa_0 \ll Y$ to be

$$\chi_0 = \frac{-\kappa_0 \; \mu \; YG/4}{\mu Y + 2\kappa_0 \mu - \kappa_0 Y} x^2 y^2 + P(x) + Q(y) \tag{22}$$

$$A_0 = \frac{\mu \; YG}{\mu Y + 2\kappa_0 \mu - \kappa_0 Y} xy \tag{23}$$

These functions do not satisfy the boundary conditions and are corrected by an additional exponentially growing term inside the boundary layer whose thickness scales with κ_1.

1.2.2 Inside the boundary layer

The working equations are the following

$$\partial_1 \partial_2 \chi = -\kappa_0 A + \kappa_1 \triangle A \tag{24}$$

$$\frac{1}{Y} \triangle^2 \chi - \frac{1}{\mu} \partial_1^2 \partial_2^2 \chi = -G + \partial_1 \partial_2 A \tag{25}$$

Near boundary $x = L$, the y derivatives are small and x derivatives are large. Introducing an internal variable, $\eta = (L - x)/\delta$, the equations become,

$$-\delta \partial_\eta \partial_2 \chi = -\delta^2 \kappa_0 A + \kappa_1 \left(\partial_\eta^2 + \delta^2 \partial_2^2 \right) A \tag{26}$$

$$-\delta^3 \partial_\eta \partial_2 A = \frac{1}{Y} \left(\partial_\eta^4 + \delta^4 \partial_2^4 + 2\delta^2 \partial_\eta^2 \partial_2^2 \right) \chi \tag{27}$$

$$- \frac{1}{\mu} \delta^2 \partial_\eta^2 \partial_2^2 \chi \tag{28}$$

If the χ and A are sufficiently slowly varying in y near $x = L$, then

$$\begin{aligned} \delta^2 \kappa_0/\kappa_1 A &= \partial_\eta^2 A && (29)\\ -Y\delta^3 \partial_\eta \partial_2 A &= \partial_\eta^4 \chi && (30)\end{aligned}$$

the solutions are of the following form

$$\begin{aligned} A(x,y) &= \frac{\mu\, YG}{\mu Y + 2\kappa_0\mu - \kappa_0 Y} xy + f(y)\, \left(e^{-(L-x)/l_{shear}} - e^{-(L+x)/l_{shear}}\right) \\ &+ g(x)\, \left(e^{-(L-y)/l_{shear}} - e^{-(L+y)/l_{shear}}\right) && (31)\end{aligned}$$

where $\delta = l_{shear} = \sqrt{\kappa_1/\kappa_0}$. Correspondingly, χ is

$$\begin{aligned} \chi(x,y) &= \frac{-\kappa_0\, \mu\, YG/4}{\mu Y + 2\kappa_0\mu - \kappa_0 Y} x^2 y^2 + p(x^2+y^2) \\ &- Yl_{shear}^3 \Bigg(f'(y)\, \left(e^{-(L-x)/l_{shear}} + e^{-(L+x)/l_{shear}}\right) \\ &\quad + g'(x)\, \left(e^{-(L-y)/l_{shear}} + e^{-(L+y)/l_{shear}}\right)\Bigg) && (32)\end{aligned}$$

For 31 and 32, to satisfy 24 and **??**, the forms of $f(x)$ and $g(y)$ are

$$\begin{aligned} f(y) &= a_1 y \\ g(x) &= a_1 x \end{aligned}$$

We determine the constants a_1 and p by satisfying the boundary condition in the limit $\kappa_1 \to 0$ and $\kappa_0 G \to 0$. Thus,

$$\begin{aligned} a_1 &= \frac{\mu Y G l_{shear} \left(1 + \tanh\left(L\sqrt{\frac{\kappa_0}{\kappa_1}}\right)\right)}{2(\mu Y + 2\kappa_0\mu - \kappa_0 Y)} \\ p &= \frac{\kappa_0\, \mu\, YG}{4(\mu Y + 2\kappa_0\mu - \kappa_0 Y)} L^2 \end{aligned}$$

Hence, the fields simplify to the following final expressions

$$\begin{aligned} A(x,y) &= \frac{\mu Y G}{\mu Y + 2\kappa_0\mu - \kappa_0 Y}\left(xy + l_{shear}\frac{x\, \sinh(y/l_{shear}) + y\, \sinh(x/l_{shear})}{\cosh(L/l_{shear})}\right) \\ \chi(x,y) &= \frac{\mu Y G/4}{\mu Y + 2\kappa_0\mu - \kappa_0 Y}\Bigg(L^2(x^2+y^2) - x^2y^2 \\ &\quad -4Yl_{shear}^4\, \frac{\cosh(x/l_{shear}) + \cosh(y/l_{shear})}{\cosh(L/l_{shear})}\Bigg) \end{aligned}$$

where, we have small parameters κ_1 and κ_0 satisfying $l_{shear} \ll L$ and $GL^2\kappa_0/Y \to 0$. Thus $l_{shear} \ll \sqrt{Y/G\kappa_0}$, which gives $\kappa_1/Y \ll 1/G$ or $G \ll Y/\kappa_1$. Hence, this solution is valid in this limit of sufficiently weakly curved substrate.

2 Numerics

The metamaterials are modeled as linear and torsional spring networks with each spring having energy

$$E_{lin} = \frac{k}{2}\left(\frac{l^2 - \bar{l}^2}{2\bar{l}^2}\right)^2, \tag{33}$$

where l is the spring length, $\bar{l}$ the equilibrium length, and k the spring constant. For small strains, $l \approx \bar{l} + \delta l$ and the energy reduces to

$$E_{lin} \approx \frac{k}{2}\left(\frac{l}{\bar{l}}\right) \tag{34}$$

Torsional springs are modeled as

$$E_{tor} = \frac{K}{2}\left(\theta - \bar{\theta}\right)^2, \tag{35}$$

where θ is the angle between a pair of edges and $\bar{\theta}$ is their equilibrium angle, and K is the torsional spring constant.

Finally, each geometrically-confined vertex is modeled with energy

$$E_{con} = \frac{k_{con}}{2}\left(z - h(x,y)\right)^2, \tag{36}$$

where $h(x,y)$ is the height function of a surface and (x,y,z) are the coordinates of the vertex. The constraint k_{con} is set to 100 times the largest modulus and we have confirmed that k_{con} is always sufficiently large so as to not change the numerical results.

The total energy was minimized with the standard L-BFGS algorithm and conjugate gradient algorithm as implemented by Mathematica version 12.

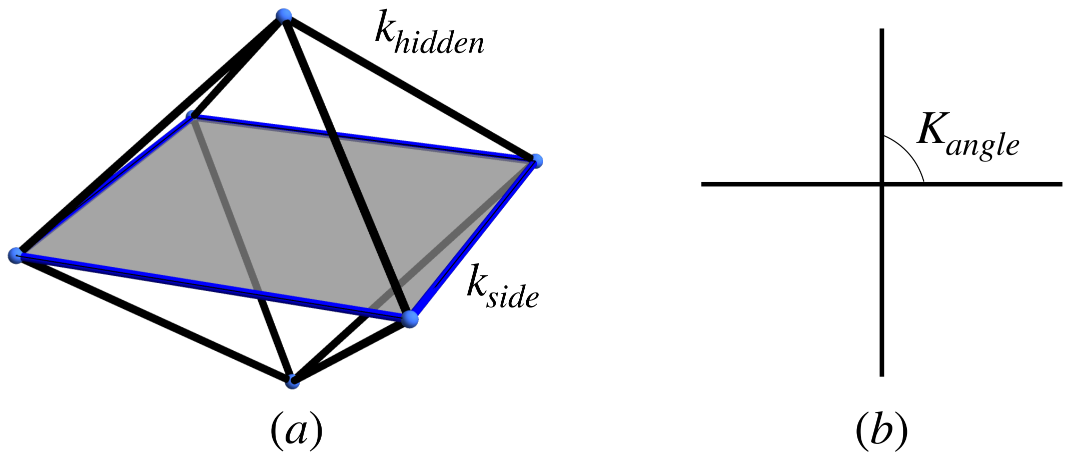

Figure 1 (a) One unit of the dilational metamaterial. (b) A vertex of the shear metamaterial.

2.1 Dilational metamaterial

The dilational metamaterial is constructed as a system of counter-rotating squares. Each square is actually modeled according to Fig. 1, with hidden edges leading to a vertex above and below the centroid of the square. The sides of the square have spring constant $k_{side} = 1$ and $k_{hidden} = 10^{-3}$ sets the bending rigidity of the square. Torsional springs are placed at the vertices where squares meet.

2.2 Shear metamaterial

The shear metamaterial is built from a square lattice of springs with spring constant $k = 1$. There is a torsional spring with spring constant K_{bend} preventing the bending of each rod through the vertex and an additional torsional spring with spring constant K_{angle} at one corner of each square of the metamaterial.