Analytical description of elastocapillary membranes held by needles
Electronic Supplementary Material

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1 Basic definitions

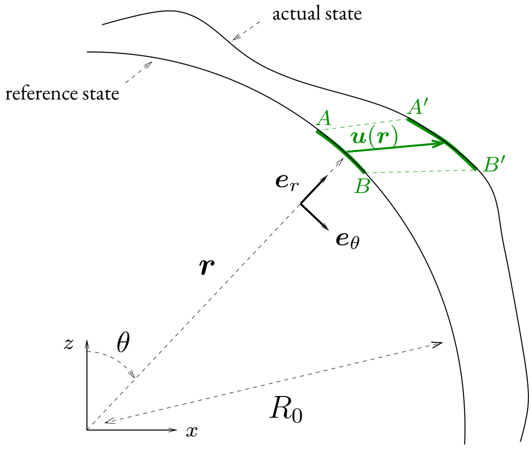

Figure S1: Mathematical definitions associated to the displacement field $\boldsymbol{u}(\boldsymbol{r})$ (in green). The reference state is a spherical cap of radius R_0. The unit vectors $(\boldsymbol{e}_r, \boldsymbol{e}_\theta)$ are associated to the reference state position. The "axisymmetric torsionless deformations" (ATD) hypothesis amounts to write $\boldsymbol{u}(\boldsymbol{r}) = u_r(r,\theta)\boldsymbol{e}_r + u_\theta(r,\theta)\boldsymbol{e}_\theta$ for a 3D elastic capsule (see text). For an approximate 2D description of an elastocapillary membrane, the formula for the displacement of surface elements will be analogous, but for u_r and u_θ which are independent of r.

1.1 Kinematics of axisymmetric torsionless deformations

An elastic membrane is solid and its physical state description requires to record the actual displacement of any physical element from its reference state. As a consequence (and in contrast with a liquid interface), the physical states of an elastic membrane are ultimately described in terms of the displacement field $\boldsymbol{u}(\boldsymbol{r}) = \boldsymbol{r}_{\text{eq}}(\boldsymbol{r}) - \boldsymbol{r}$, where $\boldsymbol{r}$ is the position of any physical element of the membrane *in the reference state*, and $\boldsymbol{r}_{\text{eq}}$ is the actual position of this element in the equilibrium state considered.

The initial spherical shape invites a description of $\boldsymbol{u}$ in terms of spherical coordinates, as sketched in Fig. S1. Furthermore, we assume axisymmetric and torsionless deformations (ATD). This precludes the description of buckled states where the Oz rotational symmetry is spontanously broken. For any physical element of the reference state at position $\boldsymbol{r} = R_0\boldsymbol{e}_r$, the ATD hypothesis allows to write $\boldsymbol{u}(\boldsymbol{r}) = u_r(r,\theta)\boldsymbol{e}_r + u_\theta(r,\theta)\boldsymbol{e}_\theta$. Notice that both u_r and u_θ keep a dependence with respect to r, because strictly speaking, a thin elastic membrane is a 3D material with a finite thickness. A correct effective bidimensional description of it is possible, but is necessarily built upon the 3D elasticity description of the material elements. Notice also that $\boldsymbol{u}$ keeps an azimuthal (ϕ) dependence via the directions of the unit vectors $(\boldsymbol{e}_r, \boldsymbol{e}_\theta)$.

The energy cost for a deformation of a material element of the 3D membrane at $\boldsymbol{r}$ (in green in Fig. S1) is associated to the relative scalar displacements of infinitesimally close points. These displacements are entirely accounted for by the so-called strain tensor $[\varepsilon_{3D}]$ [Audoly and Pomeau(2010)], which relates the squared distance $\boldsymbol{A'B'}^2$ in the actual configuration to the squared distance $\boldsymbol{AB}^2$ of two points A and B infinitesimally close to each other in the reference state (see fig. S1).

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Let A and B be two material points of this extended elastic or elastocapillary material. Writing $\boldsymbol{AB} = \mathrm{d}\boldsymbol{r}$ (and a similar primed expression), the intrinsic (base independent) relation

$$(\mathrm{d}\boldsymbol{r}')^2 - (\mathrm{d}\boldsymbol{r})^2 = 2(\mathrm{d}\boldsymbol{r})^T[\varepsilon_{3D}](\mathrm{d}\boldsymbol{r}) \tag{ESI 1}$$

actually defines the strain tensor (with the additional constraint of being symmetric) via a quadratic form. If now the spherical coordinates (r, θ, ϕ) are chosen to parametrize $\boldsymbol{r}$, and the local spherical frame $(\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi)$ to express $[\varepsilon]$, the different elements ε_{ij} of the tensor (with $i, j \in \{r, \theta, \phi\}$) can be written in terms of $(u_{3\mathrm{D},r}, u_{3\mathrm{D},\theta}, u_{3\mathrm{D},\phi})$ (defined by $\boldsymbol{u}_{3\mathrm{D}}(\boldsymbol{r}) = u_{3\mathrm{D},r}\boldsymbol{e}_r + u_{3\mathrm{D},\theta}\boldsymbol{e}_\theta + u_{3\mathrm{D},\phi}\boldsymbol{e}_\phi$). Within the ATD hypothesis, $\varepsilon_{r\phi} = \varepsilon_{\theta\phi} = 0$, and only four tensor elements are nonzero and independent (the matrix is symmetric). Actually, in the following, we will have use only of $\varepsilon_{\theta\theta}$ and $\varepsilon_{\phi\phi}$ (and, noticeably, not of ε_{rr}), whose expressions are

$$\varepsilon_{\phi\phi} = \varepsilon_\phi + \frac{1}{2}\varepsilon_\phi^2 \tag{ESI 2}$$

$$\varepsilon_{\theta\theta} = \varepsilon_\theta + \frac{1}{2}\varepsilon_\theta^2 + \frac{\psi^2}{2} \tag{ESI 3}$$

where

$$\varepsilon_\phi = [u_r + u_\theta \cot\theta]/r \tag{ESI 4}$$

$$\varepsilon_\theta = [u_r + u_\theta']/r \tag{ESI 5}$$

$$\psi = [u_r' - u_\theta]/r, \tag{ESI 6}$$

and $u' = \mathrm{d}u/\mathrm{d}\theta$. The quantities termed ε_θ and ε_ϕ are the linear parts of $\varepsilon_{\theta\theta}$ and $\varepsilon_{\phi\phi}$ respectively. ψ is in the small $\boldsymbol{u}$ regime the rotation angle of the surface normal along a meridian. Notice that conversely, for a spherical cap shaped thin membrane, $r \equiv R_0$ and the displacement fields are in one-to-one correspondence with the strain tensor $\varepsilon_{\theta,\phi}$ via the formulas

$$\begin{aligned} u_\theta/R_0 &= -\sin\theta \int_\theta^{\theta_n} \mathrm{d}\theta' \frac{[\varepsilon_\theta - \varepsilon_\phi]_{(\theta')}}{\sin\theta'} \\ u_r/R_0 &= \varepsilon_\phi + \cos\theta \int_\theta^{\theta_n} \mathrm{d}\theta' \frac{[\varepsilon_\theta - \varepsilon_\phi]_{(\theta')}}{\sin\theta'}. \end{aligned} \tag{ESI 7}$$

The particular difficulty of the elasticity is that the fully linear-in-$\boldsymbol{u}$ theory is in general unable to describe the physics for an elastic body firmly clamped somewhere. Föppl and von Karmán [Audoly and Pomeau(2010)] succeeded in giving the proper minimal theory accounting for such situations : It consists in discarding any non linear term in $\varepsilon_{\phi\phi}$ and $\varepsilon_{\theta\theta}$ but the $\psi^2/2$ in $\varepsilon_{\theta\theta}$, because for an elastic membrane the rotation angle ψ becomes non analytic and of the same order of magnitude as u_θ/R_0 in the vicinity of the clamping boundary conditions.

2 From 3D Hookean elastic energy to quasi-2D thin incompressible membrane within the ATD

Even if we are to confine the description to small deformations where all elastic modellings converge to the Hookean limit, let assume temporarily that the elastic part of the surface can be described by a Neo-Hookean incompressible material [Bouzidi and Van(2004)] : For such a material, the 3D elastic energy density is simply given by

$$e_{\mathrm{el}} = G \cdot \mathrm{tr}(\varepsilon_{3\mathrm{D}}), \tag{ESI 8}$$

where the shear modulus G is connected to the Young modulus E by $G = E/3$ for an incompressible material. This relation is complemented by the incompressibility relation

$$\det(1 + 2\varepsilon_{3\mathrm{D}}) - 1 = 2\mathrm{tr}(\varepsilon_{3\mathrm{D}}) + 2([\mathrm{tr}(\varepsilon_{3\mathrm{D}})]^2 - \mathrm{tr}(\varepsilon_{3\mathrm{D}}^2)) + 4\det(\varepsilon_{3\mathrm{D}}) = 0. \tag{ESI 9}$$

This relation, which relates different orders in $\varepsilon_{3\mathrm{D}}$ shows that the trace of $\varepsilon_{3\mathrm{D}}$ is actually not $O(\varepsilon_{3\mathrm{D}})$ (in spite of the appearances), but is rather of order $\varepsilon_{3\mathrm{D}}^2$.

To reduce the description of a fully 3D material to a thin sheet of thickness t (in the reference state) described by an effective 2D description, another hypothesis is needed to reduce the number of equations. Within the membrane theory, the key point is that the absence of external longitudinal stress imposes locally to the membrane forces the vanishing of the on-surface shear strain :$\sigma_{r,\theta} = \sigma_{r,\phi} \simeq 0$ at the contact between membrane and air (a consequence of the third Newton law). The small thickness of the membrane allows to assume this condition to be valid inside the material as well. In our case, this condition amounts to assuming $\varepsilon_{r\theta} = \varepsilon_{r\phi} = 0$ on every point of the elastic membrane. To see this, we remind the reader of the relation between stress and strain tensor elements in the incompressible limit : $\sigma_{i,j} = \frac{E}{3}[\delta_{i,j} - \mathrm{Cof}(1+2\varepsilon)_{ij}]$, where $\mathrm{Cof}(X)$ is the cofactor matrix of X. Writing from this formula that $\sigma_{r\theta} = \sigma_{r\phi} = 0$ everywhere within the membrane (the so-called "membrane approximation") reads

$$\begin{pmatrix} 2[2\varepsilon_{\phi\phi}+1] & -4\varepsilon_{\theta\phi} \\ 4\varepsilon_{\theta\phi} & -2(1+2\varepsilon_{\theta\theta}) \end{pmatrix} \begin{pmatrix} \varepsilon_{r\theta} \\ \varepsilon_{r\phi} \end{pmatrix} = 0. \tag{ESI 10}$$

In the limit of small ε, this leads to $\varepsilon_{r\theta} = \varepsilon_{r\phi} = 0$ everywhere. (see [Audoly and Pomeau(2010)] for further details). If one has an inspection of the proper 3D expression of $\varepsilon_{r\theta}$, namely

$$2\varepsilon_{r\theta} = \psi + \partial_r u_\theta + \psi \partial_r u_r + \varepsilon_\theta \partial_r u_\theta, \tag{ESI 11}$$

one sees that this cancellation amounts to have $\partial_r u_\theta$ equating $-\psi$ everywhere at the linear order, or the vanishing of the more complicated above expression if the linear order is not sufficient. As for $\varepsilon_{r\phi}$, it is exactly zero in the ATD hypothesis. As a result, equation (ESI 9) becomes a simple relation between diagonal elements of $[\varepsilon]$ and the effective Neo-Hookean energy *surface* density can be cast as

$$e_{\mathrm{el}}^{2d} = 2Gt \frac{\varepsilon_{\theta\theta}^2 + \varepsilon_{\phi\phi}^2 + \varepsilon_{\theta\theta}\varepsilon_{\phi\phi} + 2\varepsilon_{\theta\theta}\varepsilon_{\phi\phi}[\varepsilon_{\theta\theta} + \varepsilon_{\phi\phi}]}{(1+2\varepsilon_{\theta\theta})(1+2\varepsilon_{\phi\phi})} \simeq 2Gt \left(\varepsilon_{\theta\theta}^2 + \varepsilon_{\phi\phi}^2 + \varepsilon_{\theta\theta}\varepsilon_{\phi\phi} \right). \tag{ESI 12}$$

The total elastic energy reads thus in the so-called Hookean limit

$$E_{\mathrm{el}} \simeq 4\pi G t R_0^2 \int_0^{\theta_e} \mathrm{d}\theta \sin\theta \left(\varepsilon_{\theta\theta}^2 + \varepsilon_{\phi\phi}^2 + \varepsilon_{\theta\theta}\varepsilon_{\phi\phi} \right), \tag{ESI 13}$$

where $\varepsilon_{\theta\theta}$ and $\varepsilon_{\phi\phi}$ are given by the expressions (ESI 2)-(ESI 6), where the parameter r becomes the constant R_0, and where $(\varepsilon_\theta, \varepsilon_\phi, \psi)$ do depend only on θ, as u_r and u_θ in this effective 2D description. The last expression is the quadratic approximation in ε, but is nevertheless not quadratic in $\boldsymbol{u}_{2\mathrm{D}} = u_r(\theta)\boldsymbol{e}_r + u_\theta(\theta)\boldsymbol{e}_\theta$ because of the geometric non-linearity in the relation between ε and u : As stressed in [Audoly and Pomeau(2010)], the non linearities of elasticity come from material properties *and* geometrical nonlinearities, rendering elastic problems beyond small departures from equilibrium quite hard to analyse in general.

3 Variations of the bubbloon volume

The variations upon deformations of the bubbloon internal volume V_i are given by the exact formula

$$\delta V_i = \delta \left(\frac{1}{3} \int_{\mathrm{r.s.}} \mathrm{d}\boldsymbol{S}_{\mathrm{mod}} \cdot (\boldsymbol{r} + \boldsymbol{u}(\boldsymbol{r})) \right), \tag{ESI 14}$$

where the integral is over the reference sphere, $\mathrm{d}\boldsymbol{S}_{\mathrm{mod}}$ is the modified surface element vector after deformation. Using the spherical coordinates, one has $\mathrm{d}\boldsymbol{S}_{\mathrm{modified}} = \mathrm{d}\theta \mathrm{d}\phi \partial_\theta (r\boldsymbol{e}_r + \boldsymbol{u}) \times \partial_\phi (r\boldsymbol{e}_r + \boldsymbol{u})$, and after a calculation, for a ATD deformation from a spherical reference shape of radius R_0, we obtain

$$\delta V_i = \delta \left(\frac{2R_0^3\pi}{3} \int_0^{\theta_n} \mathrm{d}\theta \sin\theta (1+\varepsilon_\phi) \left((1+\frac{u_r}{R_0})(1+\varepsilon_\theta) - \frac{u_\theta}{R_0}\psi \right) \right). \tag{ESI 15}$$

Using the fact that $\psi = \varepsilon_\phi' + (\varepsilon_\phi - \varepsilon_\theta)\cot\theta$, and after a integration par parts and noting that the integrated term has a null variation, we obtain that $\delta V_i = \delta \widetilde{V_i}$, where

$$\widetilde{V_i} = \frac{2R_0^3\pi}{3} \int_0^{\theta_n} \mathrm{d}\theta \sin\theta (1+\varepsilon_\phi)^2 \left(1 + \frac{3}{2}\varepsilon_\theta - \frac{1}{2}\varepsilon_\phi \right). \tag{ESI 16}$$

Therefore, the volume variations can be expressed in terms of the strain tensor elements only.

4 Full Routh Hamiltonian

The structure of the full Lagrangian (Eq. (14)

of the paper) allows to give an explicit expression for the Hamiltonian, despite the fact that the mapping $\varepsilon'_\phi = \varepsilon'_\phi(p, \varepsilon_\phi)$ is only implicit. This is because on the one side, the Lagrange equation for Δ_ε is

$$\left.\frac{\partial \mathcal{L}}{\partial \Delta_\varepsilon}\right|_{\varepsilon_\phi, \varepsilon'_\phi, \theta} = \left.\frac{\partial \mathcal{L}}{\partial \varepsilon_{\theta\theta}}\right|_{\varepsilon_\phi, \Delta_\varepsilon \theta} \times (1 + \varepsilon_\phi + \Delta_\varepsilon - \psi \cot\theta) + \left.\frac{\partial \mathcal{L}}{\partial \Delta_\varepsilon}\right|_{\varepsilon_\phi, \varepsilon_{\theta\theta}, \theta} = 0, \tag{ESI 17}$$

where on the other side we have the momentum

$$p_\phi = \left.\frac{\partial \mathcal{L}}{\partial \varepsilon'_\phi}\right|_{\varepsilon_\phi, \Delta_\varepsilon, \theta} = \left.\frac{\partial \mathcal{L}}{\partial \varepsilon_{\theta\theta}}\right|_{\varepsilon_\phi, \Delta_\varepsilon \theta} \times \psi, \tag{ESI 18}$$

hence we obtain for the modified momentum $\widetilde{p}_\phi = p_\phi / \sin\theta$

$$\widetilde{p}_\phi = \frac{\psi(1+\xi)(1+\varepsilon_\phi)^2}{1 + \varepsilon_\phi + \Delta_\varepsilon - \psi\cot\theta} \Leftrightarrow \psi = \frac{\widetilde{p}_\phi(1 + \varepsilon_\phi + \Delta_\varepsilon)}{(1+\xi)(1+\varepsilon_\phi)^2 + \widetilde{p}_\phi \cot\theta}. \tag{ESI 19}$$

We obtain the exact Routh Hamiltonian

$$\mathcal{D} = (1+\xi)(1+\varepsilon_\phi)^2 + \widetilde{p}_\phi \cot\theta, \tag{ESI 20}$$

$$\begin{aligned}\frac{\mathcal{R}}{\sin\theta}(\varepsilon_\phi, p_\phi, \Delta_\varepsilon, \theta) = {} & \frac{1 + \varepsilon_\phi + \Delta_\varepsilon}{\mathcal{D}} \left[\widetilde{p}_\phi^2 - (1+\varepsilon_\phi)\sqrt{\mathcal{D}^2 + \widetilde{p}_\phi^2}\right], \\ & + \Delta_\varepsilon \mathcal{D} - \frac{2\alpha}{3}[\varepsilon_{\theta\theta}^2 + \varepsilon_{\phi\phi}^2 + \varepsilon_{\theta\theta}\varepsilon_{\phi\phi}] + \frac{2}{3}(1+\xi)(1+\varepsilon_\phi)^3.\end{aligned} \tag{ESI 21}$$

It can be checked that a second order expansion of this latter expression gives back the quadratic approximation of the Routh Hamiltonian (Eqs. (16-17) of the article, and below) and $\widetilde{p}_\phi \sim \psi$ at this order. Another check can be done for a pure bubble, i.e. $\alpha = 0$. In this case, one finds that ($\varepsilon_\phi = (1+\xi)^{-1} - 1, p_\phi = 0, \Delta_\varepsilon = 0$) is a solution of the Routh-Hamilton equations of motion, despite the involved structure of the equations. Note that in this case, the boundary condition $\varepsilon_\phi(\theta_n) = 0$ is not enforced, due to the fully liquid nature of the interface.

5 Solution for the quadratic approximation

The quadratic approximation of the full Routh Hamiltonian (ESI 21) is $\mathcal{R}_{\text{quad}}$, given by (see also Eqs. (16-17) of the main text)

$$\frac{\mathcal{R}_{\text{quad}}}{\sin\theta} = \frac{\mathcal{R}^{(2)}}{\sin\theta} - \frac{2\alpha}{3}\left[\Delta_\varepsilon - \frac{3}{4\alpha}W\right]^2, \tag{ESI 22}$$

$$\frac{\mathcal{R}^{(2)}}{\sin\theta} \equiv \frac{p_\phi^2}{2\sin^2\theta} + (1-2\alpha)\varepsilon_\phi^2 + 2\xi\varepsilon_\phi + \frac{3}{8\alpha}W^2, \tag{ESI 23}$$

$$W \equiv p_\phi \frac{\cot\theta}{\sin\theta} + (1-2\alpha)\varepsilon_\phi + \xi. \tag{ESI 24}$$

$\mathcal{R}_{\text{quad}}$ is simplified by performing the canonical change of variables

$$\begin{aligned} Q &= \varepsilon_\phi + \frac{3}{2\alpha}\frac{p_\phi \cot\theta / \sin\theta}{1 + 3/(2\alpha)} + \frac{\xi}{1-2\alpha}, \\ P &= p_\phi. \end{aligned} \tag{ESI 25}$$

To anticipate slightly, the physical significance of these coordinates is given by $P = \psi\sin\theta$ and the fact that Q is affinely related to $\varepsilon_\phi + \varepsilon_\theta$. This canonical change of variables is subtended by the generating function

$$G_3(Q, p_\phi, \theta) = -Qp_\phi + p_\phi \frac{\xi}{1-2\alpha} + \frac{p_\phi^2}{2}\frac{\cot\theta}{\sin\theta}\frac{1}{1+2\alpha/3}, \tag{ESI 26}$$

and the equations $(P = -\partial_Q G_3, \varepsilon_\phi = -\partial_{p_\phi} G_3)$ [Goldstein(1980)]. According to the Hamiltonian theory [Goldstein(1980)], $\mathcal{R}^{(2)}$ has to be replaced by $\mathcal{K}^{(2)}$ defined by (we disregard irrelevant constants)

$$\frac{\mathcal{K}^{(2)}}{\sin\theta} \equiv \omega_\psi^2 \frac{P^2}{2\sin^2\theta} + \frac{\frac{1}{2}-\alpha}{\omega_\psi^2}\frac{Q^2}{2}, \qquad \text{(ESI 27)}$$

$$\omega_\psi^2 = \frac{1}{1+\frac{3}{2\alpha}}. \qquad \text{(ESI 28)}$$

Consequently, $\mathcal{R}_{\text{quad}}$ is replaced by $\mathcal{K}_{\text{quad}} = \mathcal{K}^{(2)} - \frac{2\alpha}{3}[\Delta_\varepsilon - \frac{3}{4\alpha}W]^2$ where now $W = Q(1-2\alpha) + 4\omega_\psi^2 P\frac{\cot\theta}{\sin\theta}$. The dynamical equations for (Q, P) are the ordinary Hamilton equations $\{Q' = \partial_P \mathcal{K}_{\text{quad}}, P' = -\partial_Q \mathcal{K}_{\text{quad}}\}$, supplemented by the Routh equation $\partial_{\Delta_\varepsilon}\mathcal{K}_{\text{quad}} = 0 \Leftrightarrow \Delta_\varepsilon = 3W/(4\alpha)$. The combination of the Hamilton equations leads to two autonomous equations,

$$\begin{aligned} Q'' + Q'\cot\theta + (\tfrac{1}{2}-\alpha)Q = 0, \\ P'' - P'\cot\theta + (\tfrac{1}{2}-\alpha)P = 0, \end{aligned} \qquad \text{(ESI 29)}$$

and the last is equivalent to Eq. (18)

in the main text for ψ. A direct inspection of the DLMF repository [DLMF()] (Section 14.2) gives the solution for ψ (Eqs. (19-20) of the main text) in terms of the Legendre (Ferrers) functions

$$\psi = \psi_n P_\nu^1(\cos\theta)/P_\nu^1(\cos\theta_n), \qquad \text{(ESI 30)}$$

$$\nu \equiv -\frac{1}{2} + \sqrt{\frac{3}{4} - \alpha}, \qquad \text{(ESI 31)}$$

(notice $\psi_n = \psi(\theta_n)$). For Q, we obtain from [DLMF()] the solution $Q = \omega_\psi^2 \psi_n P_\nu(\cos\theta)/P_\nu^1(\cos\theta_n)$, where the constant in front of $P_\nu(\cos\theta)$ has been obtained from the Hamilton equation for Q and the fact that [DLMF()] $P_\nu^1(y) = -(1-y^2)^{1/2}\mathrm{d}P_\nu(y)/dy$. We can also write down the explicit solutions for ε_ϕ and $\varepsilon_\theta - \varepsilon_\phi$

$$\varepsilon_\phi = -\frac{\xi}{1-2\alpha} + \left(P_\nu(\cos\theta) - \frac{3\cot\theta}{2\alpha}P_\nu^1(\cos\theta)\right)\frac{\omega_\psi^2\psi_n}{P_\nu^1(\cos\theta_n)}, \qquad \text{(ESI 32)}$$

$$\varepsilon_\theta - \varepsilon_\phi = -\frac{3\omega_\psi^2\psi_n}{2\alpha P_\nu^1(\cos\theta_n)}P_\nu^2(\cos\theta), \qquad \text{(ESI 33)}$$

(where the last expression comes from the formula (14.10.1) of [DLMF()]). As $(P_\nu(\cos\theta)/P_\nu^1(\cos\theta))\tan\theta \sim 2/(\alpha-\frac{1}{2})$ near $\theta = 0$, we have $\varepsilon_\theta = \varepsilon_\phi$ at the apex, as expected. This boundary condition is thus forcefully imposed by the structure of the equations. Moreover, one has

$$\varepsilon_\phi + \varepsilon_\theta = \left(\frac{3}{4\alpha} + \frac{1}{2}\right)Q - \frac{2\xi}{1-2\alpha}, \qquad \text{(ESI 34)}$$

which shows that Q is affinely related to $\varepsilon_\theta + \varepsilon_\phi$.

From the expression (ESI 33) of Δ_ε and (ESI 7), and using (14.6.1) of [DLMF()], we obtain the following solutions for the displacement field

$$\begin{aligned} u_\theta/R_0 &= \frac{3\omega_\psi^2\psi_n}{2\alpha}\left(\frac{\sin\theta}{\sin\theta_n} - \frac{P_\nu^1(\cos\theta)}{P_\nu^1(\cos\theta_n)}\right), \\ u_r/R_0 &= -\frac{\xi}{1-2\alpha} + \psi_n\omega_\psi^2\left(-\frac{3}{2\alpha}\frac{\cos\theta}{\sin\theta_n} + \frac{P_\nu(\cos\theta)}{P_\nu^1(\cos\theta_n)}\right). \end{aligned} \qquad \text{(ESI 35)}$$

6 The susceptibility χ

In this section, we derive an explicit expression for the asphericity parameter

$$\chi = \frac{\cos\theta_n}{\cos\theta_n - 1}\frac{\delta H_a}{\delta R_e}, \qquad \text{(ESI 36)}$$

where H_a is the base-to-apex vertical distance and R_e is the equatorial radius (half the maximal width of the bubbloon) as shown in Fig. 1

of the main text. The variation is understood as resulting from a small pressure difference ξ. The prefactor ensures that $\chi = 1$ for a pure bubble, irrespective of its shape. From the shape equation $\boldsymbol{OM} = (R_0 + u_r)\boldsymbol{e_r} + u_\theta \boldsymbol{e_\theta}$ and the Eq.s (ESI 7), one obtains simply

$$\chi = \frac{\cos\theta_n}{\cos\theta_n - 1}\frac{u_r(0)}{u_r(\pi/2)} = \frac{P_\nu^1(\cos\theta_n) - \dfrac{2\alpha}{3}\cot\left(\frac{\theta_n}{2}\right)(1 - P_\nu(\cos\theta_n))}{P_\nu^1(\cos\theta_n) + \dfrac{2\alpha}{3}\tan(\theta_n)(P_\nu(0) - P_\nu(\cos\theta_n))}. \tag{ESI 37}$$

(ESI 38)

When $\alpha = 0$, one recovers $\chi = 1$ whatever the value of θ_n. This susceptibility parameter is shown in Fig. 4 (main text) for various values of θ_n.

7 Elastic limit

For a pure elastic membrane, the relevant Lagrangian

$$\frac{\mathcal{L}_{\text{el}}}{\sin\theta} = \varepsilon_{\theta\theta}^2 + \varepsilon_\phi^2 + \varepsilon_{\theta\theta}\varepsilon_\phi - \zeta(\varepsilon_\phi + \varepsilon_\theta), \tag{ESI 39}$$

$$\zeta \equiv \frac{R_0\Delta, P}{4Gt} \tag{ESI 40}$$

is obtained from Eq. (14) (main text) by multiplication by $\gamma/2Gt$, taking the limit $\gamma \to 0$, and neglecting all higher order terms proportional to ζ. In this expression, one takes $\varepsilon_{\theta\theta} = \varepsilon_\theta + \psi^2/2$ according to the Föppl-von Karman theory of elastic capsules [Audoly and Pomeau(2010)] : the purely quadratic approximation does not give a solution compatible with the boundary conditions (which explains that whereas $\varepsilon_{\phi\phi}$ is linearized to ε_ϕ in the above Lagrangian, this is not the case for $\varepsilon_{\theta\theta}$). The Hamiltonian $\mathcal{H}_{\text{el}}$ is given, with $\widetilde{p} = p/\sin\theta = \zeta\psi/(1 - \psi\cot\theta)$, *at the quadratic order*, by

$$\frac{\mathcal{H}_{\text{el}}(\varepsilon_\phi, p)}{\sin\theta} = \frac{\widetilde{p}^2}{2\zeta} - \frac{3}{2}\widetilde{p}\varepsilon_\phi\cot\theta + \frac{1}{4}(\zeta + \widetilde{p}\cot\theta)^2 + \frac{\varepsilon_\phi\zeta}{2} - \frac{3}{4}\varepsilon_\phi^2, \tag{ESI 41}$$

hence, keeping only the lowest order terms, we obtain the linear equation for ψ

$$\psi'' + \psi'\cot\theta - \psi\left[\frac{3}{2\zeta} - \frac{1}{2} + \cot^2\theta\right] = 0, \tag{ESI 42}$$

with a solution $\psi(\theta) \propto P^1_{-\frac{1}{2}+i\tau}(\cos\theta)$, where $\tau = \sqrt{3/(2\zeta) - 7/4} \sim \sqrt{3/2\zeta}$. The surprising fact that a non quadratic theory boils down to a linear differential equation (or equivalently, to a quadratic Hamiltonian), comes from the fact that ψ is of order $\sqrt{\zeta}$, therefore Eq. (ESI 42) is a linear equation for the observable preempting the linear order. In this limit, the susceptibility χ (defined in Eq. (25) (main text)) is $\cos\theta_n/(\cos\theta_n - 1)$.

8 Beyond the quadratic theory

For θ_n around the critical angle $\theta_n^\star$, the quadratic theory fails (see main paper), and one expects $\varepsilon_\phi, \varepsilon_\theta, p_\phi = O(\sqrt{\xi})$ instead of $O(\xi)$. At precisely $\theta_n = \theta_n^\star$, the boundary condition $\varepsilon_\phi(\theta_n) = 0$ is verified by the quadratic theory whatever the value of $\psi_n = \psi(\theta_n)$ (see main text, Eq. (25)) , and we show below that ψ_n is actually determined by the higher order of the expansion.

From (ESI 21), we first define and compute the lowest order $O(|\boldsymbol{u}|^3)$ in the displacement field $\boldsymbol{u}$ in $[\mathcal{K} - \mathcal{K}_{\text{quad}}]_{(Q,P,\theta,\Delta_\varepsilon)}$ (where $\mathcal{K} = \mathcal{R} + \partial_\theta G_3$)

$$\frac{\mathcal{K}}{\sin\theta} - \frac{\mathcal{K}_{\text{quad}}}{\sin\theta} = \frac{\mathcal{K}^{(3)}}{\sin\theta} + o(|\boldsymbol{u}|)^3, \tag{ESI 43}$$

$$\frac{\mathcal{K}^{(3)}}{\sin\theta}(Q, P, \theta, \Delta_\varepsilon) \equiv (\frac{P^2}{2\sin^2\theta} + \varepsilon_\phi^2)(\varepsilon_\theta - \varepsilon_\phi) + \frac{2}{3}\varepsilon_\phi^3 - \frac{2\alpha}{3}[(\varepsilon_\theta + \tfrac{\varepsilon_\phi}{2})(\varepsilon_\theta^2 + \frac{P^2}{\sin^2\theta}) + (\varepsilon_\phi + \tfrac{\varepsilon_\theta}{2})\varepsilon_\phi^2]. \tag{ESI 44}$$

Notice that we have already implicitly performed the first canonical change of variables $(\varepsilon_\phi, p_\phi) \to (Q, P)$, given by Eq. (ESI 25). Hence, $\varepsilon_\phi = \varepsilon_\phi(Q, P, \theta)$ is given in (ESI 44) by Eq. (ESI 25) and $\varepsilon_\theta = \varepsilon_\phi + \Delta_\varepsilon$. Now, we seek yet another canonical transformation in order to make $\mathcal{K}^{(2)}$ (defined in Eq. (ESI 27)) totally disappear from the new Routh Hamiltonian (It is important to stress that the Routh theory is compatible with the usual canonical change of variables, with the proviso that the latter does not involve the Routh variables (here Δ_ε). This explains why the canonical perturbation theory is not intended to reduce $\mathcal{K}_{\text{quad}}$ as a whole, but only $\mathcal{K}^{(2)}$). To this end, we seek now a solution $S(Q, X, \theta)$ of the associated Hamilton-Jacobi equation

$$\frac{\partial S}{\partial \theta} + \mathcal{K}^{(2)}(Q, \frac{\partial S}{\partial Q}, \theta) = 0. \tag{ESI 45}$$

We assume that a solution with separate variables $S = \omega_\psi^{-2} \sin(\theta) S_{\text{r}}(\theta, X) Q^2/2$ can be found, which gives for S_{r} the equation

$$\partial_\theta S_{\text{r}} + (\cot\theta) S_{\text{r}} + S_{\text{r}}^2 + \frac{1}{2} - \alpha = 0. \tag{ESI 46}$$

This is a Riccati equation which is solved by the Ansatz $S_{\text{r}} = s'(\theta)/s(\theta)$. A family of solutions, parametrized with a constant X reads

$$S(Q, X, \theta) = \frac{Q^2}{2\omega_\psi^2} \frac{\sin\theta [P_\nu^1(\cos\theta) + X Q_\nu^1(\cos\theta)]}{P_\nu(\cos\theta) + X Q_\nu(\cos\theta)}. \tag{ESI 47}$$

The idea of the canonical expansion is now to use $S(Q, X, \theta)$ as a G_1-generating function [Goldstein(1980)] to exhibit a canonical change of variables $(Q, P) \to (X, Y)$ for which $\mathcal{K}^{(2)}$ is replaced by zero (meaning that X and Y are independent from θ at the quadratic level). The canonical change of variables reads explicitely

$$\begin{aligned} Y &= -\partial_X S = \frac{Q^2}{2\omega_\psi^2} \frac{1}{[P_\nu(\cos\theta) + X Q_\nu(\cos\theta)]^2} \Leftrightarrow Q = \pm\sqrt{2Y\omega_\psi^2}[P_\nu(\cos(\theta)) + X Q_\nu(\cos\theta)], \\ P &= \partial_Q S = \pm\sqrt{\frac{2Y}{\omega_\psi^2}} \sin\theta [P_\nu^1(\cos\theta) + X Q_\nu^1(\cos\theta)]. \end{aligned} \tag{ESI 48}$$

The new Routh Hamiltonian associated to this change of variables is

$$\frac{\Delta\mathcal{K}}{\sin\theta}(X, Y, \Delta_\varepsilon) = \frac{\mathcal{K}}{\sin\theta} - \frac{\mathcal{K}^{(2)}}{\sin\theta} = \frac{\mathcal{K}^{(3)}}{\sin\theta} - \frac{2\alpha^\star}{3}\left[\Delta_\varepsilon - \frac{3}{4\alpha^\star} W\right]^2 + o(|u|^3), \tag{ESI 49}$$

where the terms of order larger than the third have been disregarded in the last expression. Notice that the Routhian term with Δ_ε (which is formally second order) is not eliminated by the procedure because $\mathcal{K}^{(2)}(Q, P)$ has been defined from $\mathcal{R}^{(2)}$ by explicitely disregarding this term. This Routhian term actually disappears in the third order expansion, since $\Delta_\varepsilon \simeq \frac{3}{4\alpha^\star} W$ up to a term at most $O(\xi)$: Consequently, the bracketed term in the right hand side of Eq. (ESI 49) is at most of order ξ^2 and hence negligible (because the displacement field is described up to the order $\xi^{3/2}$).

The principle of the canonical (singular) perturbation theory is made clear by noting that by neglecting also the third order in $|\boldsymbol{u}|$, the Hamilton equations for X and Y would be $\partial_\theta X = \partial_\theta Y = 0$: X and Y are the canonically conjugated constants of the quadratic approximation. Moreover, in this case, we have necessarily $X = 0$ because $Q_\nu(\cos\theta)$ diverges at $\theta = 0$, which is not physically allowed (see for instance Eq. (ESI 34)). As for the constant value Y_{quad} of Y at the quadratic level, its value is imposed by the boundary condition $\varepsilon_\phi(\theta_n) = 0$, which, using Eq. (ESI 25), gives

$$\begin{aligned} \sqrt{Y_{\text{quad}}} &= \pm\frac{\xi}{\sqrt{2\omega_\psi^2(1-2\alpha)\mathcal{P}_n}}, \\ \mathcal{P}_n &= P_\nu(\cos\theta_n) - \frac{3}{2\alpha}\cot(\theta_n) P_\nu^1(\cos\theta_n). \end{aligned} \tag{ESI 50}$$

We therefore recover the problem summarized in Fig. 2 of the main text : For $\alpha < 1/2$, there exists a critical angle $\theta_n^\star(\alpha)$ for which $\mathcal{P}_n$ vanishes, and the strict quadratic theory fails. The strategy to cure the problem consists in considering minimally the variations of X and Y under the Hamiltonian $\mathcal{K}^{(3)}$, which is done by the following steps:

- We assume first that $\mathcal{P}_n$ is small, scaling $\propto \sqrt{\xi}$.
- The complete solution of the problem at the cubic order is given by Eq. (ESI 48), where X and Y are functions of θ, solutions of the Hamilton equations

$$X'(\theta) = \partial_Y \mathcal{K}^{(3)} \qquad Y'(\theta) = -\partial_X \mathcal{K}^{(3)}, \tag{ESI 51}$$

- At the apex, the boundary condition is $X(0) = 0$ because the Legendre functions $Q_\nu^{0,1}(1) = \infty$. From Eq.s (ESI 25) and (ESI 48), one sees that $X_n \equiv X(\theta_n)$ and $Y_n \equiv Y(\theta_n)$ are linked by the clamping condition $\varepsilon_\phi(\theta_n) = 0$ at the needle, which reads

$$\begin{aligned} \pm\frac{\xi}{1-2\alpha} &= \sqrt{2Y_n\omega_\psi^2}\,[\mathcal{P}_n + X_n\mathcal{Q}_n]\,, \\ \mathcal{Q}_n &\equiv Q_\nu(\cos\theta_n) - \frac{3}{2\alpha}Q_\nu^1(\cos\theta_n)\cot\theta_n, \end{aligned} \tag{ESI 52}$$

- At precisely $\theta_n = \theta_n^\star$, $\mathcal{P}_n = 0$, so the preceding equation shows that one can expect $X_n \sim \sqrt{Y_n} \sim O(\sqrt{\xi})$. As a consequence, it can be noted that in Eq. (ESI 48), the first two orders of expansion of Q and P are given.
- With this in mind, an inspection of $\mathcal{K}^{(3)}$ (Eq. (ESI 44)), and the use of Eq.s (ESI 25), (ESI 24), (ESI 48) and the fact that $\Delta_\varepsilon = 3W/4\alpha$ show that the lowest order of $\partial_Y\mathcal{K}^{(3)}$ (i) is obtained by making $X = 0$ (as well as *explicit* occurrences of ξ) in the formula and (ii) is proportional to $\sqrt{Y}$. This yields

$$X'(\theta) \simeq \pm\frac{3}{2}\sqrt{Y}\mathcal{K}^{(3)}_{(X=0,Y=1,\xi=0,\theta)}. \tag{ESI 53}$$

- For $\partial_X\mathcal{K}^{(3)}$, an inspection shows that it is actually $\propto Y^{3/2}$, therefore negligible at the order of $O(Y') = \xi$. As a result, we can assume simply that $\sqrt{Y} \simeq \sqrt{Y_n} =$ constant! Consequently, we have

$$X_n \simeq \pm\frac{3\sqrt{Y_n}}{2}\int_0^{\theta_n} \mathrm{d}\theta' \mathcal{K}^{(3)}_{(X=0,Y=1,\xi=0,\theta')} \underset{\text{def}}{\equiv} I_n\sqrt{Y_n}. \tag{ESI 54}$$

- From this, the lowest approximation for Y_n is obtained from Eq. (ESI 52) by substitution and resolution of a second order polynomial. The result depends on the signs of ξ and $\mathcal{P}_n I_n \mathcal{Q}_n$. This last quantity is $\propto \theta_n - \theta_n^\star$ for θ_n close to $\theta_n^\star$. One has to pay attention to the fact that the solution has to (i) be positive and (ii) be $O(\xi)$ for vanishing ξ at nonzero $\mathcal{P}_n$, and (iii) have a correct $O(\sqrt{|\xi|})$ for $\mathcal{P}_n \to 0$. The constraints (i) and (ii) yield

$$\sqrt{Y_n} = \mathrm{sgn}(\xi\mathcal{P}_n)\left[-\frac{\mathcal{P}_n}{2I_n\mathcal{Q}_n} + \frac{\mathcal{P}_n}{2I_n\mathcal{Q}_n}\sqrt{1 + \frac{4\xi I_n\mathcal{Q}_n}{\mathcal{P}_n^2(1-2\alpha)\sqrt{2\omega_\psi^2}}}\right]. \tag{ESI 55}$$

  A problem shows up when considering the constraint (iii): the formula (ESI 55) has a correct real and positive limit for $\mathcal{P}_n \to 0$ only if $\xi I_n\mathcal{Q}_n > 0$, which reads

$$\sqrt{Y_n} \underset{\theta_n \simeq \theta_n^\star}{\simeq} \sqrt{\frac{\xi}{I_n\mathcal{Q}_n(1-2\alpha)\sqrt{2\omega_\psi^2}}}. \tag{ESI 56}$$

  To summarize, in the close vicinity of $\theta_n^\star$, <u>the sign of ξ is constrained</u>.

- In the expression ψ_n, we have to take care that a subdominant contribution has to be taken into account from Eq. (ESI 19) beyond the leading term $\psi_n \sim P(\theta_n)/\sin\theta_n$. We obtain

$$\psi_n = \mathrm{sgn}\,(\xi\mathcal{P}_n)\sqrt{\frac{2Y_n}{\omega_\psi^2}}P_\nu^1(\cos\theta_n) + (\sqrt{Y_n})^2\left[\sqrt{\frac{2}{\omega_\psi^2}}I_nQ_\nu^1(\cos\theta_n) + \frac{2}{\omega_\psi^2}\left(\frac{3}{4\alpha} - 1\right)\cot(\theta_n)[P_\nu^1(\cos\theta_n)]^2\right]. \tag{ESI 57}$$

A mathematically rigorous description of $\psi_n(\xi, \theta_n)$ in the vicinity of $\theta_n = \theta_n^\star$ requires to assume the scaling $\theta_n - \theta_n^\star = O(\sqrt{\xi})$, define $\Theta = (\theta_n - \theta_n^\star)/s\sqrt{\xi}$ (s is well-chosen constant) and write $\mathcal{P}_n \sim \mathfrak{p}(\theta_n - \theta_n^\star)$, because $\mathcal{P}_n(\theta)$ vanishes precisely at $\theta_n^\star$. Taking into account only the dominant $\sqrt{\xi}$ order, and choosing $s = [4I_n\mathcal{Q}_n/(\mathfrak{p}^2(1-2\alpha)\sqrt{2\omega_\psi^2})]^{1/2}$, one obtains (assuming $I_n\mathcal{Q}_n > 0$ for simplicity)

$$\frac{\psi_n}{\sqrt{\xi}}(\Theta) \simeq M \operatorname{sgn}\left(\left.\frac{\mathrm{d}\mathcal{P}_n}{\mathrm{d}\theta}\right|_{\theta_n^\star} \right) \left[-\Theta + \Theta\sqrt{1 + \frac{1}{\Theta^2}} \right], \tag{ESI 58}$$

$$M = |\psi_n/\sqrt{\xi}|_{\max} = \frac{2^{1/4}}{\omega_\psi^{3/2}} \frac{P_\nu^1(\cos\theta_n)}{\sqrt{(1-2\alpha)I_n\mathcal{Q}_n}}. \tag{ESI 59}$$

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