SUPPLYMENETRY INFORMATION

Spectral Non-Hermitian quantization lineshape controlled by phonon dressing in various phases of Eu^{3+}: $BiPO_4$

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SI. Analytical Calculations for the Non-Hermitian Quantization or Exceptional points control:

The second-order Fluorescence (FL) system is given as;

$$\rho_{FL}^{(2)} = \frac{|G_1|^2}{(\Gamma_{02} + i\Delta_1 + |G_1|^2/(\Gamma_{00} + 2i\Delta_1) + |G_{p1}|^2/(\Gamma_{01} + i\Delta_1 + i\Delta_{p1}))\Gamma_{22}} \quad \text{(S1)}$$

Firstly, from $\Gamma_{02} + i\Delta_1 + |G_1|^2/(\Gamma_{00} + 2i\Delta_1)$ the fraction Δ_1 can have two values. Secondly, by splitting the above level, from $|G_{p1}|^2/(\Gamma_{01} + i\Delta_1 + i\Delta_{p1})$ the fraction can have two values, so Δ_1 have three values in total, $\Gamma_{20} + i\Delta_1 + \frac{|G_1|^2}{(\Gamma_{00} + 2i\Delta_1)} = 0$, $(\Gamma_{20} + i\Delta_1)(\Gamma_{00} + 2i\Delta_1) + |G_1|^2 = 0$, and $\Gamma_{20}\Gamma_{00} - i\Delta_1\Gamma_{00} + 2i\Delta_1\Gamma_{20} - 2\Delta_1^2 + |G_1|^2 = 0$.

SI.1 Real part quantization:

Substitute $\Delta_1 = a$ into the formula in the above photon dressing formulas, we get: $\Gamma_{20}\Gamma_{00} - ia\Gamma_{00} + 2ia\Gamma_{20} - 2a^2 + 4iab + |G_1|^2 = 0$. Find the root of the real part of the denominator by ignoring b^2 (reason: b^2 is very small), and we get: It can be solved for $a = \pm\sqrt{\frac{\Gamma_{20}\Gamma_{00} + |G_1|^2}{2}}$, which corresponds to linewidth. Now substitute $\Delta_1 = b$ into formula (2), we get: $-i4ab + ia(\Gamma_{20} + 2\Gamma_{00}) = 0$ $b = \frac{\Gamma_{00} - 2\Gamma_{20}}{4}$, where $b = \frac{\Gamma_{00} - 2\Gamma_{20}}{4}$ corresponding to the lifetime (Γ). Secondly Solved for: $(\Delta_1 - x_1)(\Delta_1 - x_2) = 0$, and $\Delta_1\Gamma_{01} + i\Delta_1^2 - \Delta_1 ix_1 + i\Delta_{p1}\Delta_1 - x_1\Gamma_{01} + x_1 i\Delta_1 + ix_1^2 + x_1 i\Delta_{p1} + |G_{p1}|^2 = 0$. Putting Δ_1=c and $\Delta_1^2 = c^2 + i2cd$ we get $c\Gamma_{01} + ic^2 - 2cd + i\Delta_{p1}c - x_1\Gamma_{01} + 2x_1 ic + x_1 i\Delta_{p1} + ix_1^2 + |G_{p1}|^2 = 0$. Again, Putting x1= a+ib with b=0 $c\Gamma_{01} + ic^2 - 2cd + i\Delta_{p1}c - a\Gamma_{01} + 2iac + ia\Delta_{p1} + i(a+ib)^2 + |G_{p1}|^2 = 0$, and $c^2 + (\Delta_{p1} + a)c = 0$. A further solution is $c = \frac{-(\Delta_{p1} + 2a) \pm \sqrt{(\Delta_{p1} + 2a)^2 - 4(1)(a\Delta_{p1} - b^2)}}{2}$, $c\Gamma_{01} - 2cd - a\Gamma_{01} - 2ab + |G_{p1}|^2 = 0$ $2cd = c\Gamma_{01} - 2cd - a\Gamma_{01} - 2ab + |G_{p1}|^2$, $d = \frac{c\Gamma_{01} - 2cd - a\Gamma_{01} - 2ab + |G_{p1}|^2}{2c}$.

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SI.2 Imaginary part quantization:

Substitute $\Delta_1 = ib$ in the above photon dressing formulas, we get:

$c\Gamma_{01} + ic^2 - 2cd + i\Delta_{p1}c - (ib)\Gamma_{01} + 2i(ib)c + i(ib)\Delta_{p1} + i(a+ib)^2 + |G_{p1}|^2 = 0$. Find the root of the imaginary part of the denominator a^2, ignoring (reason: a^2 very small), we get:

$c^2 + \Delta_{p1}c - b(\Gamma_{01} + b) = 0$, $c = \frac{-\Delta_{p1} \pm \sqrt{(\Delta_{p1})^2 - 4(b(\Gamma_{01}+b))}}{2}$, $c\Gamma_{01} - 2cd - (2c + \Delta_{p1} + 2a)b + |G_{p1}|^2 = 0$ $d = \frac{c\Gamma_{01} - (2c + \Delta_{p1} + 2a)b + |G_{p1}|^2}{2c}$. Secondly Solved $(\Delta_1 - x_1)(\Delta_1 - x_2) = 0$ and $\Delta_1\Gamma_{01} + i\Delta_1^2 - \Delta_1 ix_1 + i\Delta_{p1}\Delta_1 - x_1\Gamma_{01} + x_1 i\Delta_1 + ix_1^2 + x_1 i\Delta_{p1} + |G_{p1}|^2 = 0$. Put Δ_1=c and $\Delta_1^2 = c^2 + i2cd$, $c\Gamma_{01} + ic^2 - 2cd + i\Delta_{p1}c - x_1\Gamma_{01} + 2x_1 ic + x_1 i\Delta_{p1} + ix_1^2 + |G_{p1}|^2 = 0$. Putting x1=a+ib with a=0, $c\Gamma_{01} + ic^2 - 2cd + i\Delta_{p1}c - b\Gamma_{01}i\text{-}2bc\text{-}b\Delta_{p1} + i(a+ib)^2 + |G_{p1}|^2 = 0$ and $c^2 + \Delta_{p1}c + b\Gamma_{01} - b^2 = 0$. The further solution we get $c = \frac{-(\Delta_{p1}) \pm \sqrt{(\Delta_{p1})^2 - 4(b\Gamma_{01} - b^2)}}{2}$, $c\Gamma_{01} - 2cd - 2bc - b\Delta_{p1} - 2ab + |G_{p1}|^2 = 0$, $2cd = c\Gamma_{01} - 2ab - b\Delta_{p1} - 2ab + |G_{p1}|^2$, and

$$d = \frac{c\Gamma_{01} - 2ab - b\Delta_{p1} - 2ab + |G_{p1}|^2}{2c}.$$

Numerical solution: directly solve the quadratic equation in one variable with the root-finding formula, and obtain the expression in the simulation program. The EP point can be obtained by setting the value under the root sign to 0. When the square root part is zero, the real and imaginary parts will be equal that correspond to the Exception point (EP point).

SI.3 Real Non-Hermitian quantization or EP Control

$\mathrm{Re}(EP1) = \left(\sqrt{\frac{\Gamma_{20}\Gamma_{00} + |G_1|^2}{2}} - \sqrt{\frac{\Gamma_{20}\Gamma_{00} + |G_1|^2}{2}}\right)/2 = 0$, $\mathrm{Re}(EP2) = \frac{-(\Delta_{p1} + 2a)}{2}$ Where $a = \frac{\pm\sqrt{8\Gamma_{20}\Gamma_{00}}}{2}$ refer to the real part.

SI.4 Imaginary Non-Hermitian quantization or EP control

$\mathrm{Im}(EP1) = \mathrm{Im}(EP2) = \frac{-\Delta_{p1}}{2}$. Moreover, we show the summary of the Eigenvalues, Linewidth, and Exceptional points of the four kinds of dressing i.e. single, parallel, cascade, and nested in table form below.

SII. Solution of the FL Dressing Equations

II.1 Single dressing: The second-order density matrix element for the FL case via single dressing can be written as

$$\rho_{FL}^{(2)} = \frac{|G_1|^2}{(\Gamma_{20} + i\Delta_1 + |G_{p1}|^2 / (\Gamma_{10} + i\Delta_1 - i\Delta_{p1}))\Gamma_{22}} \quad \text{(S2)}$$

Solving for the real part where the eigenvalue is given as $A_1 = (-\Delta_{p1} \pm \sqrt{U_1})/2$. A further solution for the linewidth results in $\Gamma_{e1} = (\Gamma_{10} + \Gamma_{20})/2$. The imaginary part can be calculated in the above formulas which result in the eigenvalue as $B_1=(\Gamma_{10}+\Gamma_{20}\pm\sqrt{W_1})/2$. A further solution for the linewidth can be achieved in $\Delta_{e1}=(d\Delta_{p1}-\Gamma_{20}\Delta_{p1})/2d$. where $U_1=|\Delta_{p1}|^2-4\Gamma_{10}\Gamma_{20}|G_{p1}|^2$, $W_1=(\Gamma_{10}+\Gamma_{20})^2-4\Gamma_{10}\Gamma_{20}|G_{p1}|^2$.

SII.2 Parallel dressing: The second-order density matrix element for the FL case via parallel dressing can be written as

$$\rho_{FL}^{(2)}=\frac{|G_1|^2}{(\Gamma_{20}+i\Delta_1+|G_{p1}|^2/(\Gamma_{10}+i\Delta_1-i\Delta_{p1}))(\Gamma_{22}+|G_1|^2/(\Gamma_{20}+i\Delta_1))} \quad \text{(S3)}$$

The numerical solution is derived from the root formulas whose solution for the real part eigenvalue is given as $A_{21}=-(\Gamma_{22}\Gamma_{20}+|G_1|^2)/i\Gamma_{22}$, and its solution for the linewidth is given by $\Gamma_{e21}=(\Gamma_{22}\Gamma_{20}+|G_1|^2)/\Gamma_{22}$. The first EP point for a=0. A further solution for the imaginary part eigenvalue and linewidth is given as $B_{21}=-(\Gamma_{22}\Gamma_{20}+|G_1|^2)/i\Gamma_{22}$ and $\Delta_{e21}=\Gamma_{22}\Gamma_{20}+|G_1|^2/\Gamma_{22}$, respectively. For the phonon dressing, the solution for the eigenvalue and linewidth of the real part is given as $A_{22}=(-\Delta_{p1}\pm\sqrt{U_{22}})/2$ and $\Gamma_{e22}=(\Gamma_{10}+\Gamma_{20})/2$, respectively. Moreover, the solution for the eigenvalue and linewidth of the imaginary part is given as $B_{22}=(\Gamma_{10}+\Gamma_{20}\pm\sqrt{W_{22}})/2$ and $\Delta_{e22}=(d\Delta_{p1}-\Gamma_{20}\Delta_{p1})/2d$, respectively. Where $U_{22}=|\Delta_{p1}|^2-4\Gamma_{10}\Gamma_{20}|G_{p1}|^2$, $W_{22}=(\Gamma_{10}+\Gamma_{20})^2-4\Gamma_{10}\Gamma_{20}|G_{p1}|^2$.

SII.3 Cascade dressing: The second-order density matrix element for the FL case via Cascade dressing can be written as

$$\rho_{FL}^{(2)}=\frac{|G_1|^2}{(\Gamma_{02}+i\Delta_1+|G_1|^2/(\Gamma_{00}+2i\Delta_1)+|G_{p1}|^2/(\Gamma_{01}+i\Delta_1+i\Delta_{p1}))\Gamma_{22}} \quad \text{(S4)}$$

The solution for the eigenvalue of the real part is given by $A_{31}=\pm\sqrt{8\Gamma_{20}\Gamma_{00}}/2$ and gets the linewidth as $\Gamma_{e31}=(\Gamma_{00}-2\Gamma_{20})/4$. The numerical solution for the eigenvalue and linewidth of the imaginary part is given as $B_{31}=-\Delta_{p1}\pm\sqrt{W_{31}}/2$ and $\Delta_{31}=(c\Gamma_{01}+X_{31})/2c$, respectively. Further, the numerical solution of phonon dressing eigenvalue and linewidth of the real solution is given as $A_{32}=-(\Delta_{p1}+2a)\pm\sqrt{U_{32}}/2$ and $\Gamma_{e32}=c\Gamma_{01}-V_{32}/2c$, respectively. The imaginary solution of the eigenvalue and linewidth is given by

$B_{32}=-(\Delta_{p1})\pm\sqrt{W_{32}}/2$ and $\Delta_{e32}=c\Gamma_{01}+X_{32}/2c$, respectively. Where $W_{31}=\Delta_{p1}{}^2-4b(\Gamma_{01}+b)$, $U_{32}=(\Delta_{p1}+2a)^2-4(1)(a\Delta_{p1}-b^2)$, $W_{32}=(\Delta_{p1})^2-4(b\Gamma_{01}-b^2)$ and $X_{31}=(-2c+\Delta_{p1}+2a)b+|G_{p1}|^2$, $X_{32}=-2ab-b\Delta_{p1}-2ab+|G_{p1}|^2$.

SII.4 Nested dressing: The second-order density matrix element for the FL case via Nested dressing can be written as

$$\rho_{FL}^{(2)}=\frac{|G_1|^2}{(\Gamma_{20}+i\Delta_1+|G_1|^2/(\Gamma_{00}+i2\Delta_1+|G_{p1}|^2/(\Gamma_{10}+i2\Delta_1-i\Delta_{p1}))\Gamma_{22}} \quad \text{(S5)}$$

The solution for the eigenvalue of the real part is given by $A_{41}=\pm\sqrt{\Gamma_{20}\Gamma_{00}+|G_1|^2}/2$ and gets the linewidth as $\Gamma_{e41}=(-2\Gamma_{20}-\Gamma_{00})/4$. The numerical solution for the eigenvalue and linewidth of the imaginary part is given as $B_{41}=(\Delta_{p1}\pm\sqrt{W_{41}})/4$ and $\Delta_{e41}=c\Gamma_{10}+X_{41}/4c$, respectively. Further, the numerical solution of phonon dressing eigenvalue and linewidth of the real solution is given as $A_{42}=(\Delta_{p1}+4a)\pm\sqrt{U_{42}})/4$ and $\Gamma_{e42}=c\Gamma_{10}-4ab+\Gamma_{10}a+|G_{p1}|^2/4c$, respectively. The imaginary solution of the eigenvalue and linewidth is given by $B_{42}=4b-\Delta_{p1}\pm\sqrt{W_{42}}/4$ and

$\Delta_{e42} = d\Gamma_{10} - X_{42} / 4d$, respectively. Where $W_{41} = \Delta_{p1}{}^2 - 42ib^2 + i4b\Gamma_{10}$, $U_{42} = (-\Delta_{p1} - 4a)^2 + 4a(2a + \Delta_{p1})$, $W_{42} = (\Delta_{p1} - 4b)^2 - 8(2b^2 + \Delta_{p1} b)$, and $X_{41} = -4ab + 4bc + b\Delta_{p1} + |G_{p1}|^2$, $X_{42} = 4ab + \Gamma_{10} b + |G_{p1}|^2$.

Table S1 FL real and imaginary quantization of Eigenvalues and linewidth for different dressing.

Dressing	Eigen Value		Linewidth	
	Real part	Imaginary part	Real part	Imaginary part
Single	$A_1 = (-\Delta_{p1} \pm \sqrt{U_1})/2$	$B_1 = (\Gamma_{10} + \Gamma_{20} \pm \sqrt{W_1})/2$	$\Gamma_{e1} = (\Gamma_{10} + \Gamma_{20})/2$	$\Delta_{e1} = (d\Delta_{p1} - \Gamma_{20}\Delta_{p1})/2d$
$\Gamma^* = \delta_1 - \delta_2$	$\Gamma_r^* = \sqrt{U_1}$	$\Gamma_i^* = \sqrt{W_1}$		
Parallel	$A_{21} = \pm\sqrt{\Gamma_{20}^2 + \|G_1\|^2} / 2$	$B_{21} = \pm\sqrt{\Gamma_{10}^2 + \|G_{p1}\|^2} / 2$	$\Gamma_{e21} = (\Gamma_{22}\Gamma_{20} + \|G_1\|^2)/\Gamma_{22}$	$\Delta_{e21} = \Gamma_{22}\Gamma_{20} + \|G_1\|^2 / \Gamma_{22}$
$\Gamma_P^* = \delta_1 - \delta_3$	$\Gamma_r^* = \sqrt{\Gamma_{20}^2 + \|G_1\|^2}$	$\Gamma_i^* = \sqrt{\Gamma_{10}^2 + \|G_{p1}\|^2}$		
	$A_{22} = (-\Delta_{p1} \pm \sqrt{U_{22}})/2$	$B_{22} = (\Gamma_{10} + \Gamma_{20} \pm \sqrt{W_{22}})/2$	$\Gamma_{e22} = (\Gamma_{10} + \Gamma_{20})/2$	$\Delta_{e22} = (d\Delta_{p1} - \Gamma_{20}\Delta_{p1})/2d$
	$\Gamma_r^* = \sqrt{U_{22}}$	$\Gamma_i^* = \sqrt{W_{22}}$		
Cascade	$A_{31} = \pm\sqrt{8\Gamma_{20}\Gamma_{00}} / 2$	$B_{31} = -\Delta_{p1} \pm \sqrt{W_{31}} / 2$	$\Gamma_{e31} = (\Gamma_{00} - 2\Gamma_{20})/4$	$\Delta_{31} = (c\Gamma_{01} + X_{31})/2c$
$\Gamma_C^* = \delta_1 - \delta_3$	$\Gamma_r^* = \sqrt{8\Gamma_{20}\Gamma_{00}}$	$\Gamma_i^* = \sqrt{W_{31}}$		
	$A_{32} = -(\Delta_{p1} + 2a) \pm \sqrt{U_{32}} / 2$	$B_{32} = -(\Delta_{p1}) \pm \sqrt{W_{32}} / 2$	$\Gamma_{e32} = c\Gamma_{01} - V_{32} / 2c$	$\Delta_{e32} = c\Gamma_{01} + X_{32} / 2c$
	$\Gamma_r^* = \sqrt{U_{32}}$	$\Gamma_i^* = \sqrt{W_{32}}$		
Nested	$A_{41} = \pm\sqrt{\Gamma_{20}\Gamma_{00} + \|G_1\|^2} / 2$	$B_{41} = (\Delta_{p1} \pm \sqrt{W_{41}})/4$	$\Gamma_{e41} = (-2\Gamma_{20} - \Gamma_{00})/4$	$\Delta_{e41} = c\Gamma_{10} + X_{41} / 4c$
$\Gamma_N^* = \delta_1 - \delta_3$	$\Gamma_r^* = \sqrt{\Gamma_{20}\Gamma_{00} + \|G_1\|^2}$	$\Gamma_i^* = \sqrt{W_{41}} / 2$		
	$A_{42} = (\Delta_{p1} + 4a) \pm \sqrt{U_{42}})/4$	$B_{42} = 4b - \Delta_{p1} \pm \sqrt{W_{42}} / 4$	$\Gamma_{e42} = c\Gamma_{10} - 4ab + \Gamma_{10} a + \|G_{p1}\|^2 / 4c$	$\Delta_{e42} = d\Gamma_{10} - X_{42} / 4d$
	$\Gamma_r^* = \sqrt{U_{42}} / 2$	$\Gamma_i^* = \sqrt{W_{42}} / 2$		

Table S2 FL Single dressing Non-Hermitian Quantization

Dressing	Real part		Imaginary part	
	Resonant position	linewidth	Resonant position	linewidth
Single (D_{FL})	$-\Delta_{p1}$	$(\Gamma_{10} + \Gamma_{20})/2$	$(\Gamma_{10} + \Gamma_{20})/2$	$(d\Delta_{p1} - \Gamma_{20}\Delta_{p1})/2$

Table S3 FL Parallel dressing Non-Hermitian Quantization

Dressing	Real part		Imaginary part	
	Resonant position	linewidth	Resonant position	linewidth
Parallel(D_{FL})	$-(\Gamma_{22}\Gamma_{20}+\|G_1\|^2)/i\Gamma_{22}$	0	$(\Gamma_{22}\Gamma_{20}+\|G_1\|^2)/\Gamma_{22}$	0
	$c=-\Delta_{p1}/2$	$(\Gamma_{10}+\Gamma_{20})/2$	$(\Gamma_{10}+\Gamma_{20})/2$	$(d\Delta_{p1}-\Gamma_{20}\Delta_{p1})/2d$

Table S4 FL Cascade dressing Non-Hermitian Quantization

Dressing	Real part		Imaginary part	
	Resonant position	linewidth	Resonant position	linewidth
Cascade (D_{FL})	0	$(\Gamma_{00}-2\Gamma_{20})/4$	$(\Delta_{p1}-2a)/2$	$c\Gamma_{01}-2cd-a\Gamma_{01}-2ab+\|G_{p1}\|^2/2c$
	$-\Delta_{p1}/2$	$(c\Gamma_{01}-(2c+\Delta_{p1}+2a)b+\|G_{p1}\|^2)/2c$	$-\Delta_{p1}/2$	$c\Gamma_{01}-2ab-b\Delta_{p1}-2ab+\|G_{p1}\|^2/2c$

Table S5 FL Nested dressing Non-Hermitian Quantization

Dressing	Real part		Imaginary part	
	Resonant position	linewidth	Resonant position	linewidth
Nested (D_{FL})	0	$-(2\Gamma_{20}+\Gamma_{00})/4$	$(\Delta_{p1}+4a)/4$	$(c\Gamma_{10}-4ab+\Gamma_{10}a+\left\|G_{p1}\right\|^2)/4c$
	$\Delta_{p1}/4$	$(c\Gamma_{10}--4ab+4bc+b\Delta_{p1}+\left\|G_{p1}\right\|^2)/4c$	$(4b-\Delta_{p1})/4$	$(d\Gamma_{10}-4ab+\Gamma_{10}b+\left\|G_{p1}\right\|^2)/4d$

The solution of the four types of dressing equation for the spontaneous four-wave mixing (SFWM) is described in the following.

SIII. Solution of the SFWM dressing Equations

SIII.1 Single Dressing: By opening field E1, the dressed third-order density matrix element for ES/S can be written as

$$\rho_{S/AS}^{(3)} = \frac{-iG_{AS/S}G_1G_1^{`}}{(\Gamma_{20}+i\Delta_1)(\Gamma_{00}+i\delta_{AS})\left(\Gamma_{20}+i\delta_{AS/S}+i\Delta_1^{`}+|G_{p1}|^2/\left(\Gamma_{10}+i\delta_{AS/S}+i\Delta_1^{`}-i\Delta_{p1}\right)\right)} \quad \text{(S6)}$$

This system is not a loss system and has no attenuation with $\delta_{AS}=-\delta_S$ Solve for the real part where the eigenvalue is given as $a_1=\left((\Delta_{p1}-2\Delta_1^{`})\pm\sqrt{u_1}\right)/2$. A further solution for the linewidth results in $\Gamma_{e1}=(\Gamma_{20}+\Gamma_{10})/2+v_1$. The imaginary part can be calculated by substituting $\delta_{AS}=ib$ into the above formulas which results in the eigenvalue as $b_1=\left((\Gamma_{20}+\Gamma_{10})\pm\sqrt{w_1}\right)/2$. A further solution for the linewidth can be achieved in $\Delta_{e1}=\left(\Delta_{p1}-2\Delta_1\right)/2+x_1$. Where $u_1=(\Delta_{p1}-2\Delta_1^{`})^2+4(\Gamma_{20}\Gamma_{10}-\Delta_1^{`2}+\Delta_1^{`}\Delta_{p1}+|G_{p1}|^2)$ and $v_1=(\Delta_1^{`}\Gamma_{20}-\Delta_{p1}\Gamma_{20}+\Delta_1^{`}\Gamma_{10})/2a$ , $w_1=(\Gamma_{20}+\Gamma_{10})^2-4(\Gamma_{20}\Gamma_{10}-\Delta_1^{`2}+\Delta_1^{`}\Delta_{p1}+|G_{p1}|^2)$ and $x_1=(\Delta_1^{`}\Gamma_{20}-\Delta_{p1}\Gamma_{20}+\Delta_1^{`}\Gamma_{10})/2b$.

SIII.2 Parallel dressing: we got two dressing in the form of two small equations in the denominator such as G1, gama20, and GP1, gama10. Similarly, the third-order density matrix element for stocks and anti-stocks via parallel dressing can be written as

$$\rho_{AS/S}^{(3)} = \frac{-iG_{S/AS}G_1G_1^{`}}{(\Gamma_{20}+i\Delta_1)\left(\Gamma_{00}+i\delta_{S/AS}+|G_1|^2/(\Gamma_{20}+2i\delta_{S/AS})\right)\left(\Gamma_{20}+i\delta_{S/AS}+i\Delta_1^{`}+|G_{p1}|^2/\left(\Gamma_{10}+i\delta_{S/AS}+i\Delta_1^{`}-i\Delta_{p1}\right)\right)} \quad \text{(S7)}$$

The numerical solution is derived from the root formula: $\delta_{1,2}=-\left(-i(\Gamma_{20}+2\Gamma_{00})\pm\sqrt{-(\Gamma_{20}+2\Gamma_{00})^2+8\left(\Gamma_{20}\Gamma_{00}+|G_1|^2\right)}\right)/4$, Whose solution for the real part eigenvalue is given as $a_{21}=\pm\sqrt{\left(\Gamma_{20}\Gamma_{00}+|G_1|^2\right)/2}$, and its solution for the linewidth is given by $\Gamma_{e21}=(\Gamma_{20}+2\Gamma_{00})/4$ (linewidth). The first EP point for a=0. A further solution for the imaginary part eigenvalue and linewidth is given as $b_{21}=\left((\Gamma_{20}+2\Gamma_{00})\pm\sqrt{w_{21}}\right)/4$ and $\Delta_{e21}=0$, respectively. For the phonon dressing, the numerical solution is derived from the root formula: $\delta_{3,4}=-\left(-\left(i\Gamma_{20}+i\Gamma_{10}-2\Delta_1^{`}+\Delta_{p1}\right)\pm\sqrt{\left(i\Gamma_{20}+i\Gamma_{10}-2\Delta_1^{`}+\Delta_{p1}\right)^2+4\left(\Gamma_{20}\Gamma_{10}+i\Delta_1^{`}\Gamma_{20}-i\Delta_{p1}\Gamma_{20}+i\Delta_1^{`}\Gamma_{10}-\Delta_1^{`2}+\Delta_1^{`}\Delta_{p1}+|G_{p1}|^2\right)}\right)/2$, where the solution for the eigenvalue and linewidth of the real part is given as $a_{22}=\left((\Delta_{p1}-2\Delta_1^{`})\pm\sqrt{u_{22}}\right)/2$ and $\Gamma_{e22}=(\Gamma_{20}+\Gamma_{10})/2+v_{22}$ , respectively. Moreover, the solution for eigenvalue and linewidth of the imaginary part is given as $b_{22}=(\Gamma_{20}+\Gamma_{10})/2\pm\sqrt{w_{22}}$ and $\Delta_{e22}=\left(\Delta_{p1}-2\Delta_1^{`}\right)/2+x_{22}$, respectively. Where $w_{21}=(\Gamma_{20}+2\Gamma_{00})^2-8\left(\Gamma_{20}\Gamma_{00}+|G_1|^2\right)$, $u_{22}=\left(\Delta_{p1}-2\Delta_1^{`}\right)^2+4\left(\Gamma_{20}\Gamma_{10}-\Delta_1^{`2}+\Delta_1^{`}\Delta_{p1}+|G_{p1}|^2\right)$, $v_{22}=\left(\Delta_1^{`}\Gamma_{20}-\Delta_{p1}\Gamma_{20}+\Delta_1^{`}\Gamma_{10}\right)/\left(\Delta_{p1}-2\Delta_1^{`}\right)\pm\sqrt{\left(\Delta_{p1}-2\Delta_1^{`}\right)^2+4\left(\Gamma_{20}\Gamma_{10}-\Delta_1^{`2}+\Delta_1^{`}\Delta_{p1}+|G_{p1}|^2\right)}$, $w_{22}=(\Gamma_{20}+\Gamma_{10})^2-4\left(\Gamma_{20}\Gamma_{10}-\Delta_1^{`2}+\Delta_1^{`}\Delta_{p1}+|G_{p1}|^2\right)$, $x_{22}=\left(\Delta_1^{`}\Gamma_{20}-\Delta_{p1}\Gamma_{20}+\Delta_1^{`}\Gamma_{10}\right)/(\Gamma_{20}+\Gamma_{10})\pm\sqrt{(\Gamma_{20}+\Gamma_{10})^2-4\left(\Gamma_{20}\Gamma_{10}-\Delta_1^{`2}+\Delta_1^{`}\Delta_{p1}+|G_{p1}|^2\right)}$.

SIII.3 Cascade dressing: it's also got two dressings in the one small equation denominator part with G2 and gama20 in the same bracket and Gp1 and gama10 also in the same bracket. These two dressing are in the same bracket but in parallel, it is in a different bracket.

$$\rho_{S/AS}^{(3)} = \frac{-iG_{AS/S}G_1G_1'}{(\Gamma_{20}+i\Delta_1)(\Gamma_{00}+i\delta_{AS/S}+|G_1|^2/(\Gamma_{20}+i\Delta_1+i\delta_{AS/S})+|G_{p1}|^2/(\Gamma_{10}+i\delta_{AS/S}+i\Delta_{p1}))(\Gamma_{20}+i\Delta_1'+i\delta_{AS/S})} \quad (8)$$

The solution for the eigenvalue of the real part is given by $a_{31}=\left((\Delta_1'+2\Delta_1)\pm\sqrt{u_{31}}\right)/2$ and gets the linewidth as $\Gamma_{e31}=-(\Gamma_{00}+\Gamma_{20})/2+v_{31}$. The numerical solution for the eigenvalue and linewidth of the imaginary part is given as $b_{31}=\left(-(\Gamma_{00}+\Gamma_{20})\pm\sqrt{w_{31}}\right)/2$ and $\Delta_{e31}=(\Delta_1'+2\Delta_1)/2+x_{31}$, respectively. Further, the numerical solution of phonon dressing eigenvalue and linewidth of the real solution is given as $a_{32} = \left((2a + \Delta_1 + \Delta_{p1}) \pm \sqrt{u_{32}}\right)/2$ and $\Gamma_{e32} = \left((a-c)\Gamma_{10} - 2ab + 2bc - v_{32}\right)/2c$, respectively. The imaginary solution of the eigenvalue and linewidth is given by $b_{32} = \left(-(\Gamma_{10} - 2b + \Delta_1 + \Delta_{p1}) \pm \sqrt{w_{32}}\right)/2$ and $\Delta_{e32} = \left(a\Gamma_{10} - 2ab + 2ad + x_{32}\right)/2d$, respectively. Where $u_{31} = (\Delta_1' + \Delta_1 + \Delta_2)^2 + 4(\Gamma_{00}\Gamma_{20} - \Delta_1'\Delta_1 - \Delta_1'\Delta_2 + |G_2|^2)$, $v_{31} = (2\Gamma_{00}\Delta_1 + \Gamma_{20}\Delta_1')/2a$, $w_{31} = (\Gamma_{00} + \Gamma_{20})^2 - 4(\Gamma_{00}\Gamma_{20} - 2\Delta_1'\Delta_1 + |G_1|^2)$, $x_{31} = \left(2\Gamma_{00}\Delta_1 + \Gamma_{20}\Delta_1'\right)/2b$, $u_{32} = (2a + \Delta_1 + \Delta_{p1})^2 - 4(b\Gamma_{10} + a^2 + \Delta_1 + \Delta_{p1})$, $v_{32} = (\Delta_1 + \Delta_{p1})b - |G_{p1}|^2$,

$w_{32} = (\Gamma_{10} - 2b + \Delta_1 + \Delta_{p1})^2 + 4(a^2 + (\Delta_1 + \Delta_{p1} + \Gamma_{10})b$, $x_{32} = (\Delta_1 + \Delta_{p1})a - |G_{p1}|^2$,

SIII.4 Nested dressing: Gp1 and Γ_{10} are in the denominator of the G1 and Γ_{20}. Their relationship is just dividing. In cascade, their relation is just added to each other.

$$\rho_{FL}^{(2)} = \frac{|G_1|^2}{(\Gamma_{20} + i\Delta_1 + |G_1|^2/(\Gamma_{00} + i2\Delta_1 + |G_{p1}|^2/(\Gamma_{10} + i2\Delta_1 - i\Delta_{p1}))\Gamma_{22}} \quad \text{(S9)}$$

Let $\delta^2 = a^2 - b^2 + 2ab$; we get b and ignore the real part and retain the imaginary part $\Gamma_{00}\Gamma_{20} + i\Delta_2\Gamma_{00} - ia\Gamma_{00} - ia\Gamma_{20} + a\Delta_2 + b^2 - 2abi + |G_1|^2 = 0$. The subterms for the nested dressing solutions are given as $u_{41} = 4\Delta_1^2 + 4(\Gamma_{00}\Gamma_{20} - \Delta_1^2 + |G_1|^2)$, $u_{42} = (2a + \Delta_2 - \Delta_{p1})^2 - 4(b\Gamma_{10} + a^2 + \Delta_2 - \Delta_{p1})$, $v_{41} = \Delta_1(\Gamma_{00} + \Gamma_{20})/2a$ $v_{42} = -2a_s b_s + 2b_s c_s - (\Delta_2 - \Delta_{p1})b_s - |G_{p1}|^2$ $w_{41} = (\Gamma_{00} + \Gamma_{20})^2 - 4(\Gamma_{00}\Gamma_{20} - \Delta_1^2 + |G_1|^2)$, $w_{42} = (\Gamma_{10} - 2b + \Delta_2 - \Delta_{p1})^2 + 4(a^2 + (\Delta_2 - \Delta_{p1} + \Gamma_{10})b$, $x_{42} = 2a_s d_s + (\Delta_2 - \Delta_{p1})a_s - |G_{p1}|^2$.

Table S6 SFWM real and imaginary quantization of Eigenvalues and linewidth for different dressing

Dressing	**Eigen Value**		**Linewidth**	
	Real part	Imaginary part	Real part	Imaginary part
Single	$a_1 = ((\Delta_{p1} - 2\Delta_1') \pm \sqrt{u_1})/2$	$b_1 = ((\Gamma_{20} + \Gamma_{10}) \pm \sqrt{w_1})/2$	$\Gamma_{e1} = (\Gamma_{20} + \Gamma_{10})/2 + v_1$	$\Delta_{e1} = (\Delta_{p1} - 2\Delta_1)/2 + x_1$
$\Gamma_S^* = \delta_1 - \delta_2$	$\Gamma_r^* = \sqrt{u_1}$	$\Gamma_i^* = \sqrt{w_1}$		
Parallel $\Gamma_p^* = \delta_1 - \delta_3$	$a_{21} = \pm\sqrt{(\Gamma_{20}\Gamma_{00} + |G_1|^2)/2}$	$b_{21} = \left((\Gamma_{20} + 2\Gamma_{00}) \pm \sqrt{w_{21}}\right)/4$	$\Gamma_{e21} = (\Gamma_{20} + 2\Gamma_{00})/4$	$\Delta_{e21} = 0$
	$\Gamma_r^* = \sqrt{\Gamma_{20}\Gamma_{00} + |G_1|^2}$	$\Gamma_i^* = \sqrt{w_{21}}$		
	$a_{22} = \left((\Delta_{p1} - 2\Delta_1') \pm \sqrt{u_{22}}\right)/2$	$b_{22} = ((\Gamma_{20} + \Gamma_{10}) \pm \sqrt{w_{22}})/2$	$\Gamma_{e22} = (\Gamma_{20} + \Gamma_{10})/2 + v_{22}$	$\Delta_{e22} = (\Delta_{p1} - 2\Delta_1')/2 + x_{22}$
	$\Gamma_r^* = \sqrt{u_{22}}$	$\Gamma_i^* = \sqrt{w_{22}}$		
Cascade $\Gamma_C^* = \delta_1 - \delta_3$	$a_{31} = \left((\Delta_1' + 2\Delta_1) \pm \sqrt{u_{31}}\right)/2$	$b_{31} = -((\Gamma_{00} + \Gamma_{20}) \pm \sqrt{w_{31}})/2$	$\Gamma_{e31} = -(\Gamma_{00} + \Gamma_{20})/2 + v_{31}$	$\Delta_{e31} = (\Delta_1' + 2\Delta_1)/2 + x_{31}$
	$\Gamma_r^* = \sqrt{u_{31}}$	$\Gamma_i^* = \sqrt{w_{31}}$		
	$a_{32} = \left((2a + \Delta_1 + \Delta_{p1}) \pm \sqrt{u_{32}}\right)/2$	$b_{32} = \left(-(\Gamma_{10} - 2b + \Delta_1 + \Delta_{p1}) \pm \sqrt{w_{32}}\right)/2$	$\Gamma_{e32} = \left((a-c)\Gamma_{10} - 2ab + 2bc - v_{32}\right)/2c$	$\Delta_{e32} = \left(a\Gamma_{10} - 2ab + 2ad + x_{32}\right)/2d$

	$\Gamma_r^* = \sqrt{u_{32}}$	$\Gamma_i^* = \sqrt{w_{32}}$		
	$-\Delta_1^{'}$	Γ_{20}	0	0
Nested $\Gamma_N^* = \delta_1 - \delta_3$	$a_{41} = (2\Delta_1 \pm \sqrt{u_{41}})/2$	$b_{41} = ((-\Gamma_{00} - \Gamma_{20}) \pm \sqrt{w_{41}})/2$	$\Gamma_{e41} = (\Gamma_{00} + \Gamma_{20})/2 - v$	$\Delta_{e41} = \Delta_1(\Gamma_{00} + \Gamma_{20})/2b + \Delta_1$
	$\Gamma_r^* = \sqrt{u_{41}}$	$\Gamma_i^* = \sqrt{w_{41}}$		
	$a_{42} = (2a + \Delta_2 - \Delta_{p1} \pm \sqrt{u_{42}})/2$	$b_{42} = (-\Gamma_{10} + 2b - \Delta_2 + \Delta_{p1} \pm \sqrt{w_{42}})/2$	$\Gamma_{e42} = ((a-c)\Gamma_{10} + v_{42})/2c$	$\Delta_{e42} = (a\Gamma_{10} - 2ab + x_{42})/2d$
	$\Gamma_r^* = \sqrt{u_{42}}$	$\Gamma_i^* = \sqrt{w_{42}}$		
	Δ_1	Γ_{20}	0	0

Where r and i represent the real and imaginary parts.

Table S7 SFWM Single dressing Non-Hermitian Quantization

Dressing	Real part		Imaginary part	
	Resonant position	linewidth	Resonant position	width
Single (d_{AS})	$\Delta_{p1} - 2\Delta_1^{'}$	$(\Gamma_{20} + \Gamma_{10})/2 + v_1$	$(\Gamma_{20} + \Gamma_{10})/2$	$(\Delta_{p1} - 2\Delta_1)/2 + x_1$

Table S8 SFWM Parallel dressing Non-Hermitian Quantization

Dressing	Real part		Imaginary part	
	Resonant position	linewidth	Resonant position	linewidth
Parallel (d_{AS})	0	$(\Gamma_{20} + 2\Gamma_{00})/4$	$(\Gamma_{20} + 2\Gamma_{00})/4$	0
	$(\Delta_{p1} - 2\Delta_1^{'})/2$	$(\Gamma_{20} + \Gamma_{10})/2 + v_{22}$	$(\Gamma_{20} + \Gamma_{10})/2$	$(\Delta_{p1} - 2\Delta_1^{'})/2 + x_{22}$

Table S9 SFWM Cascade dressing Non-Hermitian Quantization

Dressing	Real part	Imaginary part

	Resonant position	linewidth	Resonant position	linewidth
	$(\Delta_1' + 2\Delta_1)/2$	$-(\Gamma_{00}+\Gamma_{20})/2 + v_{31}$	$-(\Gamma_{00}+\Gamma_{20})/2$	$(\Delta_1' + 2\Delta_1)/2 + x_{31}$
Cascade(d_{AS})	$(2a+\Delta_1+\Delta_{p1})/2$	$((a-c)\Gamma_{10} - 2ab + 2bc - v_{32})/2c$	$(\Gamma_{10} + 2b - \Delta_1 - \Delta_{p1})/2$	$(a\Gamma_{10} - 2ab + 2ad + x_{32})/2d$
	$-\Delta_1^{'}$	0	Γ_{20}	0

Table S10 SFWM Nested dressing Non-Hermitian Quantization

Dressing	Real part		Imaginary part	
	Resonant position	linewidth	Resonant position	linewidth
	Δ_1	$(\Gamma_{00}+\Gamma_{20})/2 - v$	$(-\Gamma_{00}-\Gamma_{20})/2$	$\Delta_1(\Gamma_{00}+\Gamma_{20})/2b + \Delta_1$
Nested (d_{AS})	$(2a+\Delta_2-\Delta_{p1})/2$	$((a-c)\Gamma_{10} + v_{42})/2c$	$(-\Gamma_{10} + 2b - \Delta_2 + \Delta_{p1})/2$	$(a\Gamma_{10} - 2ab + x_{42})/2d$
	Δ_1	0	Γ_{20}	0

Table S11 Eigenvalues of cascade three dressing for FL

Energy level	Real part	imaginary part
$\|0>$	$A_1 = -\sqrt{8\Gamma_{20}\Gamma_{00}}/2$ $A_2 = +\sqrt{8\Gamma_{20}\Gamma_{00}}/2$ $A_0 = \sqrt{U_1}$	$B_1 = -\Delta_{p1} - \sqrt{W_{31}}/2$ $B_2 = -\Delta_{p1} + \sqrt{W_{31}}/2$ $B_0 = \sqrt{W_1}$
$\|1>$	$A_3 = -(\Delta_{p1} + 2a) - \sqrt{U_{32}}/2$ $A_4 = -(\Delta_{p1} + 2a) + \sqrt{U_{32}}/2$	$B_3 = -(\Delta_{p1}) - \sqrt{W_{32}}/2$ $B_4 = -(\Delta_{p1}) + \sqrt{W_{32}}/2$