# S1. Governing equations of nonlinear simulations

In this section, we describe the dimensional governing equations for the elasto-hydrodynamics of an active filament in a viscous fluid with a prescribed viscosity gradient.

The flexible filament is modelled as an elastic rod using the classical Kirchhoff theory. Its internal elastic force at a given arc-length $\tilde{s}$, defined as $\tilde{\mathbf{f}}_{\mathrm{e}} = \tilde{\tau}\mathbf{t} + \tilde{N}\mathbf{n}$, denotes the contact force applied from the downstream part of the rod ($s' > \tilde{s}$) onto the upstream segment ($s' < \tilde{s}$). The internal bending moment $\tilde{\mathbf{m}}_{\mathrm{e}}$ is determined in the same way, capturing the moment of contact forces exerted from the downstream side to the upstream side. For a slender planar rod, the internal bending moment is $\tilde{\mathbf{m}}_{\mathrm{e}} = A\theta_{\tilde{s}}\mathbf{e}_z$, where $A$ is the bending modulus and $\mathbf{e}_z = \mathbf{e}_x \times \mathbf{e}_y$.

The torque balance for the differential element $\mathrm{d}\tilde{s}$ located between $\tilde{s}$ and $\tilde{s} + \mathrm{d}\tilde{s}$ is expressed as:

$$-\tilde{\mathbf{m}}_{\mathrm{e}}(\tilde{s}) + \tilde{\mathbf{m}}_{\mathrm{e}}(\tilde{s} + \mathrm{d}\tilde{s}) + (\mathbf{t}\mathrm{d}\tilde{s}) \times \tilde{\mathbf{f}}_{\mathrm{e}}(\tilde{s}) = \mathbf{0}, \tag{S.1}$$

leading to $\tilde{N} = -A\theta_{\tilde{s}\tilde{s}}$. Balancing the force on the same element $\mathrm{d}\tilde{s}$ gives:

$$-\tilde{\mathbf{f}}_{\mathrm{e}}(\tilde{s}) + \tilde{\mathbf{f}}_{\mathrm{e}}(\tilde{s} + \mathrm{d}\tilde{s}) + \tilde{\mathbf{f}}_{\mathrm{h}}\mathrm{d}\tilde{s} - f\mathbf{t}(\tilde{s})\mathrm{d}\tilde{s} = \mathbf{0}. \tag{S.2}$$

By substituting the expression for the hydrodynamic force density $\tilde{\mathbf{f}}_{\mathrm{h}}$ (equation 2 in the main article), the normal and tangential components of Eq. (S.2) can be obtained as:

$$\tilde{\xi}_{\perp}\mathbf{n} \cdot \frac{\partial \tilde{\mathbf{r}}}{\partial \tilde{t}} = \tilde{\tau}\theta_{\tilde{s}} - A\theta_{\tilde{s}\tilde{s}\tilde{s}}, \tag{S.3a}$$

$$\tilde{\xi}_{\parallel}\mathbf{t} \cdot \frac{\partial \tilde{\mathbf{r}}}{\partial \tilde{t}} = \tilde{\tau}_{\tilde{s}} + A\theta_{\tilde{s}}\theta_{\tilde{s}\tilde{s}} - f. \tag{S.3b}$$

Differentiating the above equations with respect to $\tilde{s}$ and using the inextensibility condition $\frac{\partial \tilde{\mathbf{r}}_{\tilde{s}}}{\partial \tilde{t}} \cdot \mathbf{t} = 0$, we derive

$$\tilde{\xi}_{\perp}\frac{\partial \theta}{\partial \tilde{t}} + A\theta_{\tilde{s}\tilde{s}\tilde{s}\tilde{s}} - (\tilde{\tau}\theta_{\tilde{s}})_{\tilde{s}} - \frac{\tilde{\xi}_{\perp}}{\tilde{\xi}_{\parallel}}\theta_{\tilde{s}}(\tilde{\tau}_{\tilde{s}} + A\theta_{\tilde{s}}\theta_{\tilde{s}\tilde{s}} - f) + \frac{\partial \tilde{\xi}_{\perp}}{\tilde{\xi}_{\perp}\partial \tilde{s}}(\tilde{\tau}\theta_{\tilde{s}} - A\theta_{\tilde{s}\tilde{s}\tilde{s}}) = 0, \tag{S.4a}$$

$$\tilde{\tau}_{\tilde{s}\tilde{s}} + (A\theta_{\tilde{s}}\theta_{\tilde{s}\tilde{s}})_{\tilde{s}} - \frac{\tilde{\xi}_{\parallel}}{\tilde{\xi}_{\perp}}\theta_{\tilde{s}}(\tilde{\tau}\theta_{\tilde{s}} - A\theta_{\tilde{s}\tilde{s}\tilde{s}}) - \frac{\partial \tilde{\xi}_{\parallel}}{\tilde{\xi}_{\parallel}\partial \tilde{s}}(\tilde{\tau}_{\tilde{s}} + A\theta_{\tilde{s}}\theta_{\tilde{s}\tilde{s}} - f) = 0. \tag{S.4b}$$

Evidently, the filament behaviour is influenced by the viscosity gradient through spatial variations in the drag coefficients $\tilde{\xi}_{\perp}$ and $\tilde{\xi}_{\parallel}$.

At the clamped base $\tilde{s} = 0$, $\theta(\tilde{s} = 0, \tilde{t}) = 0$ and $\frac{\partial \tilde{\mathbf{r}}}{\partial \tilde{t}} = \mathbf{0}$ are imposed. From equations (S.3a) and (S.3b), we obtain the following boundary conditions (BCs):

$$\tilde{s} = 0: \quad \tilde{\tau}\theta_{\tilde{s}} - A\theta_{\tilde{s}\tilde{s}\tilde{s}} = \tilde{\tau}_{\tilde{s}} + A\theta_{\tilde{s}}\theta_{\tilde{s}\tilde{s}} - f = \frac{\partial \theta}{\partial \tilde{t}} = 0. \tag{S.5}$$

The free end of filament at $\tilde{s} = L$ experiences zero force and torque, leading to

$$\tilde{s} = L: \quad \tilde{\tau} = A\theta_{\tilde{s}} = A\theta_{\tilde{s}\tilde{s}} = 0. \tag{S.6}$$

# S2. Numerical methods

This section outlines the numerical method employed to solve the dimensionless governing equations (3) along with the corresponding BCs (6) and (7) in the main text.

We utilise the Chebyshev series to discretise the filament centerline[1,2]. The differentiation operator, denoted as $D_p$, is defined as $D_p(\cdot) = \partial^p(\cdot)/\partial s^p$, where $p = 1, \ldots, 4$. For time discretization, a variant of implicit-explicit scheme[2,3] is employed. The coupling of $\tau$ and $\theta$ involves treating the higher-order spatial derivatives in nonlinear terms implicitly, while the other variables are treated explicitly. After using the backward Euler scheme for the first step, we apply an implicit-explicit scheme to solve for the $(n+1)$-th step of $[\tau, \theta]$:

$$\frac{3\theta^{n+1}-4\theta^{n}+\theta^{n-1}}{2\Delta t}+\left[D_4-\hat{\tau}D_2-2(D_1\hat{\theta})^2D_2+2\sigma D_1\right]\theta^{n+1}-3D_1\hat{\theta}D_1\tau^{n+1}+\hat{G}\left(\hat{\tau}D_1-D_3\right)\theta^{n+1}=0, \tag{S.7a}$$

$$\left[D_2-\frac{1}{2}(D_1\hat{\theta})^2\right]\tau^{n+1}+\left(D_2\hat{\theta}D_2+D_1\hat{\theta}D_3+\frac{1}{2}D_1\hat{\theta}D_3\right)\theta^{n+1}-\hat{G}\left(D_1\tau^{n+1}+D_1\hat{\theta}D_2\theta^{n+1}-\sigma\right)=0, \tag{S.7b}$$

where $\Delta t$ represents the time step, and the hat $\hat{(\cdot)}$ denotes the predicted unknowns at the next time step, approximated as $\hat{(\cdot)}=2(\cdot)^n-(\cdot)^{n-1}$.

We use 51 Chebyshev points to discretize the filament, and the time step is fixed at $\Delta t=10^{-5}$. The initial condition for $\theta$ is set to be $10^{-3}\sin(s)$, introducing a minor perturbation to initiate the potential instability. Simultaneously, the initial tension $\tau$ is assigned to the base state $\tau^{(0)}=\sigma(s-1)$.

## S3. The configuration of a point follower force

In the main article, we have focused on an elastic filament driven by a uniformly distributed follower force density along its centerline. In this section, we present the governing equation and results for another setting of active forcing: the filament is actuated by a point follower force $F$ (dimensional) at the free tip as in Refs.[4,5].

The dimensionless governing equation can be obtained by adjusting that (equation 3 in the main article) for the distributed forcing. Namely, we set the dimensionless active force density $\sigma=fL^3/A$ (for the setting of distributed forcing) to be zero in that equation, obtaining

$$\frac{\partial\theta}{\partial t}+\theta_{ssss}-(\tau\theta_s)_s-2\theta_s(\tau_s+\theta_s\theta_{ss})+G(\tau\theta_s-\theta_{sss})=0, \tag{S.8a}$$

$$\tau_{ss}+(\theta_s\theta_{ss})_s-\frac{1}{2}\theta_s(\tau\theta_s-\theta_{sss})-G(\tau_s+\theta_s\theta_{ss})=0. \tag{S.8b}$$

Besides, the dimensionless point force is defined as

$$\sigma=\frac{FL^2}{A}. \tag{S.9}$$

It is noted that for the settings of distributed and point forcing, we adopt the same notation $\sigma$ to indicate the strength of active forcing relative to elastic force. Incorporating this definition of dimensionless point force, we write the BCs:

$$s=0: \quad \tau\theta_s-\theta_{sss}=\tau_s+\theta_s\theta_{ss}=\frac{\partial\theta}{\partial t}=0, \tag{S.10a}$$

$$s=1: \quad \tau=-\sigma, \quad \theta_s=\theta_{ss}=0. \tag{S.10b}$$

The linearised governing equations are obtained following a similar approach as in the main article. The base state is characterised by $\theta^{(0)}=0$ and $\tau^{(0)}=-\sigma$. Consequently, the governing equation for $\theta^{(1)}$ and corresponding BCs read:

$$\frac{\partial\theta^{(1)}}{\partial t}+\theta^{(1)}_{ssss}+\sigma\theta^{(1)}_{ss}+G^{(0)}\left(-\sigma\theta^{(1)}_{s}-\theta^{(1)}_{sss}\right)=0, \tag{S.11a}$$

$$s=0: \quad \sigma\theta^{(1)}_s+\theta^{(1)}_{sss}=\frac{\partial\theta^{(1)}}{\partial t}=0, \tag{S.11b}$$

$$s=1: \quad \theta^{(1)}_s=\theta^{(1)}_{ss}=0. \tag{S.11c}$$

In Fig. (S1)a, the real and imaginary parts of the most unstable eigenvalues, obtained from both LSA and numerical simulations, are depicted. These values vary with the active forcing $\sigma$ and the vertical viscosity gradient $\gamma$. Fig. (S1)b illustrates that both the instability threshold $\sigma_c$ and the corresponding frequency at the onset of instability consistently decrease with $\gamma$. These trends mirror those observed in the case of distributed active force density (see Fig. 3 in the main article).

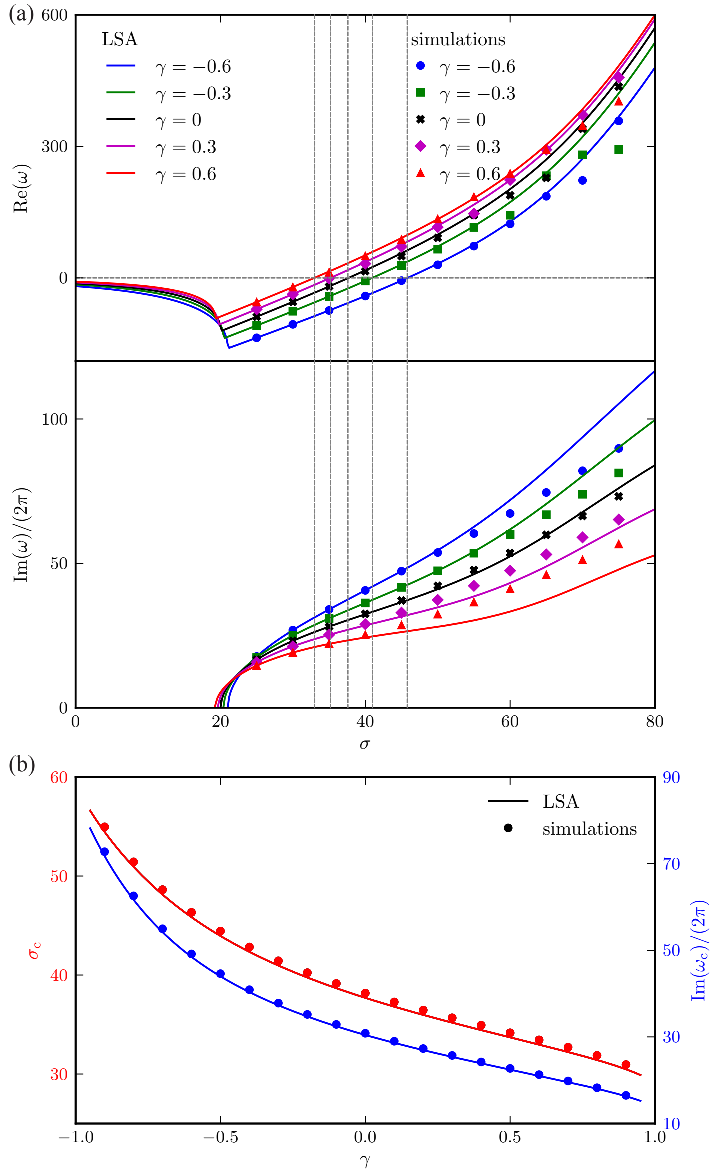


**Fig. S1** Comparison between LSA (lines) and simulations (symbols) for a filament driven by a point follower force of dimensionless strength $\sigma$. (a) Real and imaginary parts of the most unstable eigenvalues as a function of $\sigma$ for different $\gamma$. The vertical dashed lines indicate the critical value of $\sigma$ leading to instability. (b) The critical value $\sigma_c$ and oscillation frequency as a function of $\gamma$.

## S4. Instability of a filament subject to a horizontal viscosity gradient

In Fig. (S2), we present the growth rate and frequency of the most unstable eigenmodes for a filament subjected to a constant force density. These results are derived from both LSA and numerical simulations. As elucidated in the main article, our findings suggest that the influence of horizontally varying viscosity on filament instability is minimal. This observation remains valid in scenarios involving a point follower force (not shown here).

## Notes and references


[1] L. N. Trefethen, *Spectral Methods in MATLAB*, SIAM, 2000.

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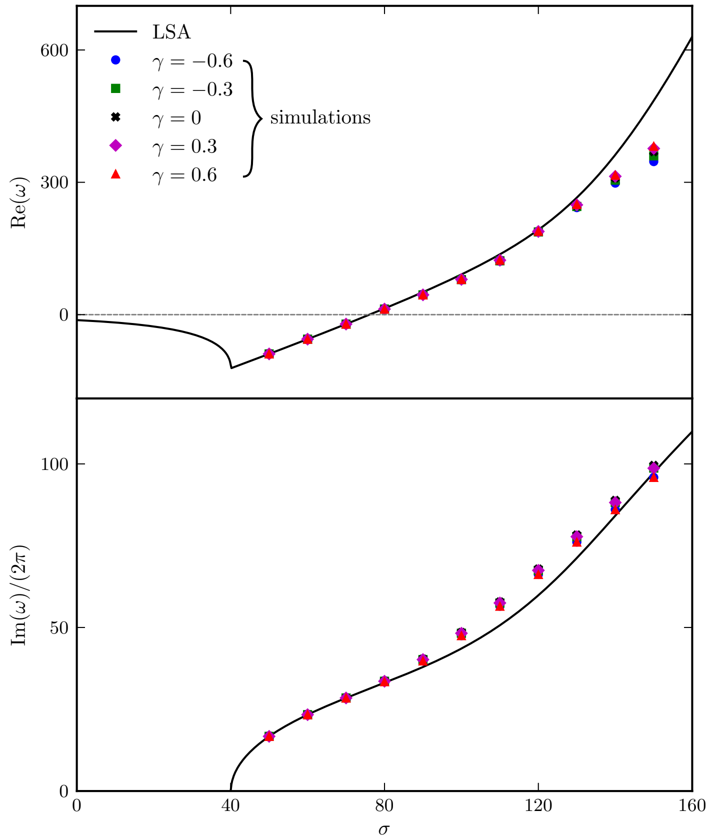


**Fig. S2** Instability of a filament driven by a distributed active force density $\sigma$ under a horizontal viscosity gradient $\gamma$. Real and imaginary parts of the most unstable eigenvalues versus $\sigma$ at different $\gamma$.

[4] G. De Canio, E. Lauga and R. E. Goldstein, *J. R. Soc. Interface*, 2017, **14**, 20170491.

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