Supplementary Information for "Phoresis Kernel Theory for Passive and Active Spheres with Nonuniform Phoretic Mobility"

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This supplementary information provides detailed calculations for the problems presented in the main manuscript. It is intended for pedagogical purposes, primarily aimed at junior researchers, and thus includes a high level of detail that may be superfluous for more advanced researchers. The derivations are crafted with this educational intent in mind, although alternative approaches could yield shorter and more concise derivations.

1. HARMONIC DRIVING FIELD

For a sphere immersed in a fluid to exhibit phoretic motion, it may be exposed to a gradient of a harmonic driving field Ψ. In the limit of a thin interaction layer, outside the interaction layer, this field obeys the Laplace equation,

$$\nabla^2\Psi = 0, \tag{1}$$

and is subjected to specific surface and far-field boundary conditions:

$$\begin{cases} \hat{\boldsymbol{n}}\cdot\nabla\Psi|_{S^+} = -\hat{\boldsymbol{n}}\cdot\boldsymbol{\Gamma}^{S^+}/\mathcal{D}, \\ \nabla\Psi|_\infty = -\boldsymbol{\Gamma}^\infty/\mathcal{D} \equiv -\Gamma^\infty\hat{\boldsymbol{e}}_z/\mathcal{D}, \end{cases} \quad \text{or equivalently,} \quad \begin{cases} \partial_r\Psi|_{S^+} = -\hat{\boldsymbol{n}}\cdot\boldsymbol{\Gamma}^{S^+}/\mathcal{D}, \\ \partial_r\Psi|_\infty = -\Gamma^\infty\cos\theta/\mathcal{D} = -r^{-1}\boldsymbol{\Gamma}^\infty\cdot\boldsymbol{x}/\mathcal{D}, \end{cases} \tag{2}$$

where S^+ represents the outer surface of the interaction layer ($r_{S^+}\approx a$), $\mathcal{D}$ is the diffusion coefficient associated with the phoretic process, and the relationship $\hat{\boldsymbol{e}}_r\cdot\hat{\boldsymbol{e}}_z = \cos\theta$ is used in the spherical coordinate system $\boldsymbol{x} = r\hat{\boldsymbol{e}}_r = (r,\theta,\phi)$.

For convenience, we divide the driving field $\Psi = \Psi^{(p)} + \Psi^{(a)}$ into the harmonic field $\Psi^{(p)}$, which accounts for the phoresis of a non-polarizable passive particle with the same geometry solely due to the far-field flux $\boldsymbol{\Gamma}^\infty$, and the harmonic field $\Psi^{(a)}$, which accounts for the self-phoresis of an active particle of the same geometry solely due to the surface flux $\boldsymbol{\Gamma}^{S^+}$. The boundary conditions for these two scenarios are

$$\begin{cases} \partial_r\Psi^{(p)}|_{S^+} = 0, \\ \partial_r\Psi^{(p)}|_\infty = -\Gamma^\infty\cos\theta/\mathcal{D} = -r^{-1}\boldsymbol{\Gamma}^\infty\cdot\boldsymbol{x}/\mathcal{D}, \end{cases} \qquad \begin{cases} \partial_r\Psi^{(a)}|_{S^+} = -\hat{\boldsymbol{n}}\cdot\boldsymbol{\Gamma}^{S^+}/\mathcal{D}, \\ \partial_r\Psi^{(a)}|_\infty = 0, \end{cases} \tag{3}$$

The general solution to the harmonic driving field $\Psi^{(\alpha)}$, expressed in terms of spherical harmonics

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$Y_{\ell m}(\theta,\phi)$, is

$$\Psi^{(\alpha)} = A_{00}^{(\alpha)} + \sum_{\ell=1}^{\infty} \sum_{-\ell\le m\le \ell} A_{\ell m}^{(\alpha)} r^{\ell} Y_{\ell m} + \sum_{\ell=0}^{\infty} \sum_{-\ell\le m\le \ell} B_{\ell m}^{(\alpha)} r^{-(\ell+1)} Y_{\ell m}, \tag{4}$$

with the radial derivative,

$$\frac{\partial \Psi^{(\alpha)}}{\partial r} = \sum_{\ell=1}^{\infty} \sum_{-\ell\le m\le \ell} \ell\, A_{\ell m}^{(\alpha)} r^{\ell-1} Y_{\ell m} + \sum_{\ell=0}^{\infty} \sum_{-\ell\le m\le \ell} [-(\ell+1)] B_{\ell m}^{(\alpha)} r^{-(\ell+2)} Y_{\ell m}. \tag{5}$$

We set $A_{00}^{(\alpha)} = 0$, or in other words, we are interested in the disturbance field. We will first obtain the solution for the passive phoresis $\Psi^{(p)}$ and then for the active self-phoresis $\Psi^{(a)}$, and add them up to obtain the harmonic driving field $\Psi = \Psi^{(p)} + \Psi^{(a)}$.

For the passive phoresis field, using $Y_{10} \propto \cos\theta$ and applying the far-field boundary condition yields $A_{10}^{(p)} Y_{10} = -\Gamma^{\infty}\cos\theta/\mathcal{D}$, $A_{1,1}^{(p)} = A_{1,-1}^{(p)} = 0$, and $A_{\ell m}^{(p)} = 0$ for $\ell \ge 2$. Then, applying the surface flux condition for the passive non-polarizable particle

$$\partial_r \Psi^{(p)}|_{r=a} = 0 = -\Gamma^{\infty}\cos\theta/\mathcal{D} + \sum_{\ell=0}^{\infty} \sum_{-\ell\le m\le \ell} [-(\ell+1)] B_{\ell m}^{(p)} a^{-(\ell+2)} Y_{\ell m} \tag{6}$$

yields $B_{0,0}^{(p)} = B_{1,1}^{(p)} = B_{1,-1}^{(p)} = 0$ and $B_{\ell m}^{(p)} = 0$ for $\ell \ge 2$. The equation $-\Gamma^{\infty}\cos\theta/\mathcal{D} - 2a^{-3} B_{10}^{(p)} Y_{10} = 0$ leads to $B_{10}^{(p)} Y_{10} = -\frac{1}{2} a^3 \Gamma^{\infty}\cos\theta/\mathcal{D}$. Thus, using $\hat{\boldsymbol{e}}_z \cdot \boldsymbol{x} = r\cos\theta$ we obtain

$$\begin{aligned}
\Psi^{(p)}(\boldsymbol{x}) &= A_{10}^{(p)} Y_{10} r + B_{10}^{(p)} r^{-2} Y_{10} \\
&= -\Gamma^{\infty} r\cos\theta/\mathcal{D} - \frac{a^3}{2r^2}\Gamma^{\infty}\cos\theta/\mathcal{D} \\
&= -\Gamma^{\infty} \hat{\boldsymbol{e}}_z \cdot \boldsymbol{x}/\mathcal{D} - \frac{a^3}{2r^3}\Gamma^{\infty} \hat{\boldsymbol{e}}_z \cdot \boldsymbol{x}/\mathcal{D} \\
&= -\boldsymbol{\Gamma}^{\infty} \cdot \boldsymbol{x}/\mathcal{D} - \frac{a^3}{2r^3}\boldsymbol{\Gamma}^{\infty} \cdot \boldsymbol{x}/\mathcal{D} \\
&= -\mathcal{D}^{-1}\left[1 + \frac{1}{2}\left(\frac{a}{r}\right)^3\right]\boldsymbol{\Gamma}^{\infty} \cdot \boldsymbol{x}
\end{aligned} \tag{7}$$

Before proceeding with the active phoresis field, we define the banana-bracket notation $\{\!\!\{\cdot\}\!\!\}_{\ell m}$ for the expansion coefficients in terms of spherical harmonics. That is, for a function $\Phi(\theta,\phi)$ defined over the surface of a sphere, we have

$$\Phi(\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \{\!\!\{\Phi\}\!\!\}_{\ell m}\, Y_{\ell,m}(\theta,\phi) \tag{8a}$$

$$\{\!\!\{\Phi\}\!\!\}_{\ell m} = \int_0^{\pi} \int_0^{2\pi} \Phi(\theta,\phi)\, Y_{\ell,m}^{*}(\theta,\phi) \sin\theta \, d\theta \, d\phi \tag{8b}$$

Now, for the active phoresis field, the far-field zero radial gradient condition $\partial_r \Psi^{(a)}|_{\infty} = 0$ necessitates $A^{(a)}_{\ell m} = 0$. The surface flux condition yields

$$\partial_r \Psi^{(a)}|_{S^+} = \sum_{\ell=0}^{\infty} \sum_{-\ell\le m\le\ell} [-(\ell+1)] B^{(a)}_{\ell\, m} a^{-(\ell+2)} Y_{\ell m} = -\hat{\boldsymbol{n}}\cdot\boldsymbol{\Gamma}^{S^+}/\mathcal{D} = -\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} \mathcal{D}^{-1} \left\{\!\!\left\{ \hat{\boldsymbol{n}}\cdot\boldsymbol{\Gamma}^{S^+} \right\}\!\!\right\}_{\ell m} Y_{\ell,m} \tag{9}$$

where, in the last step, we have expanded the surface flux $\hat{\boldsymbol{n}}\cdot\boldsymbol{\Gamma}^{S^+}$ in terms of spherical harmonics. Exploiting the orthogonality of the spherical harmonics leads to $B^{(a)}_{\ell\, m} = a^{\ell+2}(\ell+1)^{-1}\mathcal{D}^{-1}\left\{\!\!\left\{ \hat{\boldsymbol{n}}\cdot\boldsymbol{\Gamma}^{S^+} \right\}\!\!\right\}_{\ell m}$ and we obtain the active self-phoresis field,

$$\Psi^{(a)}(\boldsymbol{x}) = \mathcal{D}^{-1} \sum_{\substack{\ell=0\\ -\ell\le m\le\ell}}^{\infty} \frac{a}{\ell+1}\left(\frac{a}{r}\right)^{\ell+1} Y_{\ell,m} \left\{\!\!\left\{ \hat{\boldsymbol{n}}\cdot\boldsymbol{\Gamma}^{S^+} \right\}\!\!\right\}_{\ell m}. \tag{10}$$

Finally, we obtain the disturbance field by superposing the fields due to passive phoresis (7) and active self-phoresis (10),

$$\Psi(\boldsymbol{x}) = \Psi^{(a)}(\boldsymbol{x}) + \Psi^{(p)}(\boldsymbol{x}) = \mathcal{D}^{-1} \sum_{\substack{\ell=0\\ -\ell\le m\le\ell}}^{\infty} \frac{a}{\ell+1}\left(\frac{a}{r}\right)^{\ell+1} Y_{\ell,m} \left\{\!\!\left\{ \hat{\boldsymbol{n}}\cdot\boldsymbol{\Gamma}^{S^+} \right\}\!\!\right\}_{\ell m} - \mathcal{D}^{-1}\left[1+\frac{1}{2}\left(\frac{a}{r}\right)^3\right]\boldsymbol{x}\cdot\boldsymbol{\Gamma}^{\infty}, \tag{11}$$

which is Eq (17) in the main manuscript.

2. AXISYMMETRIC SELF-PHORETIC PARTICLE WITH MULTIPLE SURFACE REGIONS

For the general axisymmetric scenario with k axisymmetric regions, the distribution of phoretic mobility over the sphere surface is described by

$$\mu(\theta) = \begin{cases} \mu^{(1)} & 0 \le \theta < \theta_1 \\ \mu^{(2)} & \theta_1 \le \theta < \theta_2 \\ \vdots & \vdots \\ \mu^{(\mathrm{k})} & \theta_{k-1} \le \theta \le \theta_k \equiv \pi \end{cases} \tag{12}$$

where θ_i's define the boundaries of the regions. We can rewrite the expression using the Heaviside step function:

$$H(\theta-\theta') = \begin{cases} 0 & \theta < \theta' \\ 1 & \theta \ge \theta' \end{cases} \tag{13}$$

as

$$\begin{aligned} \mu(\theta) &= \mu^{(1)}[1-H(\theta-\theta_1)] + \mu^{(2)}[H(\theta-\theta_1)-H(\theta-\theta_2)] + \cdots + \mu^{(k-1)}[H(\theta-\theta_{k-2})-H(\theta-\theta_{k-1})] + \mu^{(k)}H(\theta-\theta_{k-1}) \\ &= \mu^{(1)} + [\mu^{(2)}-\mu^{(1)}]H(\theta-\theta_1) + [\mu^{(3)}-\mu^{(2)}]H(\theta-\theta_2) + \cdots + [\mu^{(k)}-\mu^{(k-1)}]H(\theta-\theta_{k-1}) \\ &= \mu^{(1)} + \sum_{j=1}^{k-1}[\mu^{(j+1)}-\mu^{(j)}]H(\theta-\theta_j) \end{aligned} \tag{14}$$

In the last term of the first line, we have omitted $-\mu^{(k)}H(\theta-\theta_k)$ to include $\theta_k = \pi$ in the calculations. Additionally, this term is non-zero in the domain $\theta > \theta_k = \pi$, which does not enter the calculations.

The motion is along the symmetry axis $\hat{\boldsymbol{e}}_z$. Using $\hat{\boldsymbol{e}}_z \cdot \hat{\boldsymbol{e}}_\theta = -\sin\theta$, we have $a\,\hat{\boldsymbol{e}}_z \cdot \nabla_S = -\sin\theta\partial_\theta$. Combining this expression with $\hat{\boldsymbol{e}}_z \cdot \hat{\boldsymbol{n}} = \hat{\boldsymbol{e}}_z \cdot \hat{\boldsymbol{e}}_r = \cos\theta$ and $\partial_\theta H(\theta-\theta') = \delta(\theta-\theta')$, where $\delta(\theta)$ is the Dirac delta function, the translational field kernel is given by

$$\begin{aligned}\hat{\boldsymbol{e}}_z \cdot \mathbb{K}_t &= -a\hat{\boldsymbol{e}}_z \cdot \nabla_S\,\mu + 2\hat{\boldsymbol{e}}_z \cdot \hat{\boldsymbol{n}}\mu \\ &= \sin\theta\,\partial_\theta\mu + 2\mu\cos\theta \\ &= \sin\theta\sum_{j=1}^{k-1}[\mu^{(j)}-\mu^{(j-1)}]\delta(\theta-\theta_{j-1}) + 2\mu^{(1)}\cos\theta + 2\cos\theta\sum_{j=1}^{k-1}[\mu^{(j)}-\mu^{(j-1)}]H(\theta-\theta_{j-1}) \\ &= 2\mu^{(1)}\cos\theta + \sum_{j=1}^{k-1}[\mu^{(j)}-\mu^{(j-1)}][\sin\theta\,\delta(\theta-\theta_{j-1}) + 2\cos\theta H(\theta-\theta_{j-1})] \end{aligned} \tag{15}$$

Defining the dimensionless function

$$\mathbb{G}(\theta;\theta_j) = \sin\theta\delta(\theta-\theta_j) + 2\cos\theta H(\theta-\theta_j), \tag{16}$$

which is Eq. (34a) in the main manuscript, we can write the field kernel for translational velocity along the symmetry axis in the form

$$\hat{\boldsymbol{e}}_z \cdot \mathbb{K}_t = 2\mu^{(1)}\cos\theta + \sum_{j=1}^{k-1}[\mu^{(j)}-\mu^{(j-1)}]\mathbb{G}(\theta;\theta_j) \tag{17}$$

which is Eq. (33a) in the main manuscript.

To obtain the flux kernel for motion along the symmetry axis,

$$\hat{\boldsymbol{e}}_z \cdot \boldsymbol{\mathcal{K}}_t^{S^+} = \sum_{\ell=0}^{\infty}\sum_{-\ell\le m\le\ell}\frac{1}{\ell+1}\,\{\!\!\{\hat{\boldsymbol{e}}_z \cdot \mathbb{K}_t\}\!\!\}_{\ell m}\,Y_{\ell,m}, \tag{18}$$

we need to obtain $\{\!\!\{\hat{\boldsymbol{e}}_z \cdot \mathbb{K}_t\}\!\!\}_{\ell m}$, and thus, $\{\!\!\{\cos\theta\}\!\!\}_{\ell m}$ and $\{\!\!\{\mathbb{G}(\theta;\theta_j)\}\!\!\}_{\ell m}$. Using $\cos\theta = AY_{1,0}(\theta,\phi)$, where $A = 2\sqrt{\pi/3}$, we have

$$\{\!\!\{\cos\theta\}\!\!\}_{\ell m} = \int_0^\pi\int_0^{2\pi} AY_{1,0}Y^*_{\ell,m}(\theta,\phi)\sin\theta\,d\theta\,d\phi = A\delta_{\ell,1}\delta_{m,0}, \tag{19}$$

leading to the first term of the flux kernel (18),

$$\sum_{\substack{\ell=0 \\ -\ell\le m\le\ell}}^{\infty}\frac{1}{\ell+1}\,\{\!\!\{2\mu^{(1)}\cos\theta\}\!\!\}_{\ell m}\,Y_{\ell,m} = 2\mu^{(1)}\sum_{\substack{\ell=0 \\ -\ell\le m\le\ell}}^{\infty}\frac{1}{\ell+1}A\delta_{\ell,1}\delta_{m,0}Y_{\ell,m} = 2\mu^{(1)}\frac{1}{2}AY_{1,0} = \mu^{(1)}\cos\theta. \tag{20}$$

Next, we calculate $\{\!\!\{\mathbb{G}(\theta;\theta_j)\}\!\!\}_{\ell m}$. Using

$$Y_{\ell,m}(\theta,\phi) = Y_{\ell,m}(\theta,0)\,e^{im\phi}, \qquad Y^*_{\ell,m}(\theta,\phi) = Y_{\ell,m}(\theta,0)\,e^{-im\phi}, \tag{21}$$

for the first term in Eq. (16), we have

$$\begin{aligned}
\{\!\!\{\sin\theta\delta(\theta-\theta_j)\}\!\!\}_{\ell m} &= \int_0^{\pi}\int_0^{2\pi}\sin\theta\delta(\theta-\theta_j)\,Y^*_{\ell,m}(\theta,\phi)\sin\theta\,d\theta\,d\phi \\
&= \int_0^{\pi}\int_0^{2\pi}\sin\theta\delta(\theta-\theta_j)\,Y_{\ell,m}(\theta,0)\,e^{-im\phi}\sin\theta\,d\theta\,d\phi \\
&= \int_0^{2\pi}e^{-im\phi}\,d\phi\int_0^{\pi}\sin^2\theta\,\delta(\theta-\theta_j)\,Y_{\ell,m}(\theta,0)\,d\theta \\
&= 2\pi\delta_{m,0}\,\sin^2\theta_j\,Y_{\ell,m}(\theta_j,0) \\
&= 2\pi\,\sin^2\theta_j\,Y_{\ell,0}(\theta_j,0)\,\delta_{m,0} \qquad (22)
\end{aligned}$$

For the second term in Eq. (16), we have

$$\begin{aligned}
\{\!\!\{2\cos\theta H(\theta-\theta_j)\}\!\!\}_{\ell m} &= \int_0^{\pi}\int_0^{2\pi}2\cos\theta H(\theta-\theta_j)\,Y^*_{\ell,m}(\theta,\phi)\sin\theta\,d\theta\,d\phi \\
&= \int_0^{\pi}\int_0^{2\pi}2\cos\theta H(\theta-\theta_j)\,Y_{\ell,m}(\theta,0)\,e^{-im\phi}\sin\theta\,d\theta\,d\phi \\
&= \int_0^{2\pi}e^{-im\phi}\,d\phi\int_0^{\pi}H(\theta-\theta_j)\,Y_{\ell,m}(\theta,0)\,[2\cos\theta\sin\theta]\,d\theta \\
&= 2\pi\delta_{m,0}\int_0^{\pi}H(\theta-\theta_j)\,Y_{\ell,m}(\theta,0)\sin2\theta\,d\theta \\
&= 2\pi\delta_{m,0}\int_{\theta_j}^{\pi}Y_{\ell,0}(\theta,0)\sin2\theta\,d\theta \qquad (23)
\end{aligned}$$

Combinig Eqns. (22) and (23) yields

$$\{\!\!\{\mathbb{G}(\theta;\theta_j)\}\!\!\}_{\ell m} = \{\!\!\{\mathbb{G}(\theta;\theta_j)\}\!\!\}_{\ell,0}\,\delta_{m,0}, \qquad (24)$$

where

$$\{\!\!\{\mathbb{G}(\theta;\theta_j)\}\!\!\}_{\ell,0} = 2\pi\sin^2\theta_j\,Y_{\ell,0}(\theta_j,0) + 2\pi\int_{\theta_j}^{\pi}Y_{\ell,0}(\theta,0)\sin2\theta\,d\theta. \qquad (25)$$

Therefore, we define

$$\begin{aligned}
\mathcal{G}(\theta;\theta_j) &= \sum_{\ell=0}^{\infty}\sum_{-\ell\le m\le\ell}\frac{1}{\ell+1}\{\!\!\{\mathbb{G}(\theta;\theta_j)\}\!\!\}_{\ell m}\,Y_{\ell,m}(\theta,\phi) \\
&= \sum_{\ell=0}^{\infty}\sum_{-\ell\le m\le\ell}\frac{1}{\ell+1}\{\!\!\{\mathbb{G}(\theta;\theta_j)\}\!\!\}_{\ell,0}\,\delta_{m,0}Y_{\ell,m}(\theta,\phi) \\
&= \sum_{\ell=0}^{\infty}\frac{1}{\ell+1}\{\!\!\{\mathbb{G}(\theta;\theta_j)\}\!\!\}_{\ell,0}\,Y_{\ell,0}(\theta,0) \\
&= \sum_{\ell=0}^{\infty}\frac{2\pi}{\ell+1}\left[Y_{\ell,0}(\theta_j,0)\,\sin^2\theta_j + \int_{\theta_j}^{\pi}Y_{\ell,0}(\theta,0)\sin2\theta\,d\theta\right]Y_{\ell,0}(\theta,0), \qquad (26)
\end{aligned}$$

which is Eq. (34b) in the main manuscript. In the last two lines $Y_{\ell,0}(\theta,0) \equiv Y_{\ell,0}(\theta,\phi)$ is written to express that the axisymmetric scenario is independent of ϕ. The flux kernel become

$$\hat{\boldsymbol{e}}_z \cdot \boldsymbol{\mathcal{K}}_t^{S^+} = \mu^{(1)} \cos\theta + \sum_{j=1}^{k-1} [\mu^{(j+1)} - \mu^{(j)}]\, \mathcal{G}(\theta;\theta_j), \tag{27}$$

which is Eq. (33b) in the main manuscript.

3. AXISYMMETRIC JANUS SWIMMER

The distribution of surface flux and phoretic mobility over the surface of the Janus swimmer is given by

$$[\hat{\boldsymbol{n}} \cdot \boldsymbol{\Gamma}^{S^+}, \mu] = \begin{cases} [\Gamma^{(1)}, \mu^{(1)}] & 0 \le \theta < \frac{\pi}{2} \\ [\Gamma^{(2)}, \mu^{(2)}] & \frac{\pi}{2} \le \theta \le \pi \end{cases} \tag{28}$$

The corresponding flux kernel (27) along the symmetry axis for this problem takes the form

$$\hat{\boldsymbol{e}}_z \cdot \boldsymbol{\mathcal{K}}_t^{S^+} = \mu^{(1)} \cos\theta + \left[\mu^{(2)} - \mu^{(1)}\right] \mathcal{G}(\theta; \tfrac{\pi}{2}). \tag{29}$$

To calculate the translational velocity, we start with the first term

$$\begin{aligned} \int_S \cos\theta\, \hat{\boldsymbol{n}} \cdot \boldsymbol{\Gamma}^{S^+} dS &= 2\pi a^2 \Gamma^{(1)} \int_0^{\pi/2} \cos\theta \sin\theta\, d\theta + 2\pi a^2 \Gamma^{(2)} \int_{\pi/2}^{\pi} \cos\theta \sin\theta\, d\theta \\ &= 2\pi a^2 \Gamma^{(1)} \times \left(\frac{1}{2}\right) + 2\pi a^2 \Gamma^{(2)} \times \left(\frac{-1}{2}\right) \\ &= \pi a^2 \left[\Gamma^{(1)} - \Gamma^{(2)}\right] \end{aligned} \tag{30}$$

For the second term, we can write

$$\mathcal{G}(\theta; \tfrac{\pi}{2}) = \sum_{\ell=0}^{\infty} \mathcal{G}_\ell(\tfrac{\pi}{2}) Y_{\ell,0}(\theta, 0), \tag{31}$$

where

$$\mathcal{G}_\ell(\pi/2) = \frac{2\pi}{\ell+1} \left[Y_{\ell,0}(\tfrac{\pi}{2}, 0) + \int_{\pi/2}^{\pi} Y_{\ell,0}(\theta, 0) \sin 2\theta\, d\theta \right]. \tag{32}$$

Therefore, the surface integration of flux weighted by $\mathcal{G}(\theta; \frac{\pi}{2})$ is

$$\begin{aligned} \int_S \mathcal{G}(\theta; \tfrac{\pi}{2})\, \hat{\boldsymbol{n}} \cdot \boldsymbol{\Gamma}^{S^+} dS &= \sum_{\ell=0}^{\infty} \mathcal{G}_\ell(\tfrac{\pi}{2}) \int_S Y_{\ell,0}(\theta, 0)\, \hat{\boldsymbol{n}} \cdot \boldsymbol{\Gamma}^{S^+} dS \\ &= 2\pi a^2 \sum_{\ell=0}^{\infty} \left[\Gamma^{(1)} \mathcal{G}_\ell(\tfrac{\pi}{2}) \int_0^{\frac{\pi}{2}} Y_{\ell,0}(\theta, 0) \sin\theta\, d\theta + \Gamma^{(2)} \mathcal{G}_\ell(\tfrac{\pi}{2}) \int_{\frac{\pi}{2}}^{\pi} Y_{\ell,0}(\theta, 0) \sin\theta\, d\theta \right] \\ &= 2\pi a^2 \Gamma^{(1)} \times \left(\frac{1}{4}\right) + 2\pi a^2 \Gamma^{(2)} \times \left(\frac{-1}{4}\right) = \frac{1}{2} \pi a^2 \left[\Gamma^{(1)} - \Gamma^{(2)}\right] \end{aligned} \tag{33}$$

ℓ	0	1	2	3	4	5	6	7	8	9	10
$\mathcal{G}_\ell(\pi/2)$	0	$\sqrt{\frac{\pi}{3}}$	$-\frac{\sqrt{5\pi}}{4}$	0	$\frac{\sqrt{\pi}}{4}$	0	$-\frac{3\sqrt{13\pi}}{64}$	0	$\frac{\sqrt{17\pi}}{32}$	0	$-\frac{35\sqrt{\frac{7\pi}{3}}}{512}$
$\int_0^{\pi/2} Y_{\ell,0}(\theta,0)\sin\theta\,d\theta$	$\frac{1}{2\sqrt{\pi}}$	$\frac{\sqrt{3/\pi}}{4}$	0	$-\frac{\sqrt{7/\pi}}{16}$	0	$\frac{\sqrt{11/\pi}}{32}$	0	$-\frac{5\sqrt{15/\pi}}{256}$	0	$\frac{7\sqrt{19/\pi}}{512}$	0
$\int_{\pi/2}^{\pi} Y_{\ell,0}(\theta,0)\sin\theta\,d\theta$	$\frac{1}{2\sqrt{\pi}}$	$-\frac{\sqrt{3/\pi}}{4}$	0	$\frac{\sqrt{7/\pi}}{16}$	0	$-\frac{\sqrt{11/\pi}}{32}$	0	$\frac{5\sqrt{15/\pi}}{256}$	0	$-\frac{7\sqrt{19/\pi}}{512}$	0

TABLE I. Terms for the integral involving second term of the flux kernel (29)

where Table I shows that only $\ell = 1$ contributes. Therefore,

$$\begin{aligned}
\hat{\boldsymbol{e}}_z \cdot \boldsymbol{U} &= \frac{-1}{4\pi a^2 \mathcal{D}} \int_{S^+} dS\, \hat{\boldsymbol{e}}_z \cdot \boldsymbol{\mathcal{K}}_t^{S^+}\, \hat{\boldsymbol{n}} \cdot \boldsymbol{\Gamma}^{S^+} \\
&= \frac{-1}{4\pi a^2 \mathcal{D}} \int_{S^+} dS\, \left\{\mu^{(1)} \cos\theta + \left[\mu^{(2)} - \mu^{(1)}\right] \mathcal{G}(\theta; \tfrac{\pi}{2})\right\} \hat{\boldsymbol{n}} \cdot \boldsymbol{\Gamma}^{S^+} \\
&= \frac{-1}{4\pi a^2 \mathcal{D}} \mu^{(1)} \int_{S^+} dS\, \cos\theta\, \hat{\boldsymbol{n}} \cdot \boldsymbol{\Gamma}^{S^+} + \frac{-1}{4\pi a^2 \mathcal{D}} \left[\mu^{(2)} - \mu^{(1)}\right] \int_{S^+} dS\, \mathcal{G}(\theta; \tfrac{\pi}{2})\, \hat{\boldsymbol{n}} \cdot \boldsymbol{\Gamma}^{S^+} \\
&= \frac{-1}{4\pi a^2 \mathcal{D}} \mu^{(1)} \pi a^2 \left[\Gamma^{(1)} - \Gamma^{(2)}\right] + \frac{-1}{4\pi a^2 \mathcal{D}} \left[\mu^{(2)} - \mu^{(1)}\right] \frac{\pi}{2} a^2 \left[\Gamma^{(1)} - \Gamma^{(2)}\right] \\
&= \frac{-1}{4\mathcal{D}} \left[\Gamma^{(1)} - \Gamma^{(2)}\right] \left\{\mu^{(1)} + \frac{1}{2}\left[\mu^{(2)} - \mu^{(1)}\right]\right\} \\
&= \frac{-1}{8\mathcal{D}} \left[\Gamma^{(1)} - \Gamma^{(2)}\right] \left[\mu^{(2)} + \mu^{(1)}\right]
\end{aligned} \tag{34}$$

which is equivalent to Eq. (37) in the main manuscript.

4. A SOURCE-SINK PROBLEM WITH A CONDUCTIVE PASSIVE REGION IN THE MIDDLE

The microswimmer consists of three regions with phoretic mobility and flux distributions,

$$\mu(\theta) = \begin{cases} \mu^{(1)} = \mu^{(a)} & 0 \le \theta < \theta_1 \\ \mu^{(2)} = \mu^{(p)} & \theta_1 \le \theta < \theta_2 \\ \mu^{(3)} = \mu^{(a)} & \theta_2 \le \theta \le \pi \end{cases}, \qquad \hat{\boldsymbol{n}} \cdot \boldsymbol{\Gamma}^{S^+} = \begin{cases} \Gamma^{(1)} = \Gamma_0 & 0 \le \theta < \theta_1 \\ \Gamma^{(2)} = 0 & \theta_1 \le \theta < \theta_2 \\ \Gamma^{(3)} = f(\Gamma_0, \theta_1, \theta_2) & \theta_2 \le \theta \le \pi \end{cases}, \tag{35}$$

respectively. The relation $\Gamma^{(3)} = f(\Gamma_0, \theta_1, \theta_2)$ is obtained via the steady-state condition for the source-sink scenario,

$$\begin{aligned}
0 = \int_S dS\, \hat{\boldsymbol{n}} \cdot \boldsymbol{\Gamma}^{S^+} &= 2\pi a^2 \int_0^\pi \hat{\boldsymbol{n}} \cdot \boldsymbol{\Gamma}^{S^+} \sin\theta\, d\theta \\
&= 2\pi a^2 \Gamma_0 \int_0^{\theta_1} \sin\theta\, d\theta + 2\pi a^2 \Gamma^{(3)} \int_{\theta_2}^{\pi} \sin\theta\, d\theta \\
&= 2\pi a^2 \Gamma_0 (1 - \cos\theta_1) + 2\pi a^2 \Gamma^{(3)} (1 + \cos\theta_2)
\end{aligned} \tag{36}$$

which leads to

$$\Gamma^{(3)} = -\frac{1-\cos\theta_1}{1+\cos\theta_2}\,\Gamma_0. \tag{37}$$

Hence, the translational flux kernel reduces to

$$\begin{aligned}\hat{\boldsymbol{e}}_z\cdot\boldsymbol{\mathcal{K}}_t^{S^+} &= \mu^{(1)}\cos\theta + \left[\mu^{(2)}-\mu^{(1)}\right]\mathcal{G}(\theta;\theta_1) + \left[\mu^{(3)}-\mu^{(2)}\right]\mathcal{G}(\theta;\theta_2)\\ &= \mu^{(a)}\cos\theta + \left[\mu^{(p)}-\mu^{(a)}\right]\mathcal{G}(\theta;\theta_1) + \left[\mu^{(a)}-\mu^{(b)}\right]\mathcal{G}(\theta;\theta_2)\\ &= \mu^{(a)}\cos\theta + \left[\mu^{(p)}-\mu^{(a)}\right]\left[\mathcal{G}(\theta;\theta_1)-\mathcal{G}(\theta;\theta_2)\right]\end{aligned} \tag{38}$$

which is Eq. (38) in the main manuscript.

If there was no passive region in the middle and the surface areas of the source and sink were equal, that is, $\theta_1 = \theta_2 = \frac{\pi}{2}$, we would have a Janus sphere, and we consider the microswimmer speed corresponding to this scenario as the reference velocity scale,

$$\begin{aligned}U_{(\mathrm{ref})} &= \frac{-1}{4\pi a^2\mathcal{D}}\int_S dS\,\hat{\boldsymbol{e}}_z\cdot\boldsymbol{\mathcal{K}}_t^{S^+}\,\hat{\boldsymbol{n}}\cdot\boldsymbol{\Gamma}^{S^+}\\ &= \frac{-1}{4\pi a^2\mathcal{D}}\left(2\pi a^2\mu^{(a)}\right)\left[\Gamma_0\int_0^{\pi/2}\cos\theta\sin\theta d\theta + (-\Gamma_0)\int_{\pi/2}^{\pi}\cos\theta\sin\theta d\theta\right]\\ &= \frac{-\mu^{(a)}}{2\mathcal{D}}\Gamma_0,\end{aligned} \tag{39}$$

which is Eq. (42) in the main manuscript.

Now, for the three-region microswimmer, we start with the uniform phoretic mobility term in the flux kernel (38),

$$\begin{aligned}\frac{-1}{4\pi a^2\mathcal{D}}\int_S dS\,\mu^{(a)}\cos\theta\,\hat{\boldsymbol{n}}\cdot\boldsymbol{\Gamma}^{S^+} &= \frac{-2\pi a^2\mu^{(a)}\Gamma_0}{4\pi a^2\mathcal{D}}\left[\int_0^{\theta_1}\cos\theta\sin\theta d\theta - \frac{1-\cos\theta_1}{1+\cos\theta_2}\int_{\theta_2}^{\pi}\cos\theta\sin\theta d\theta\right]\\ &= U_{(\mathrm{ref})}\left[\frac{1}{2}\sin^2\theta_1 - \frac{1-\cos\theta_1}{1+\cos\theta_2}\left(-\frac{1}{2}\sin^2\theta_2\right)\right]\\ &= U_{(\mathrm{ref})}\left(2+\cos\theta_1-\cos\theta_2\right)\sin^2\frac{\theta_1}{2}.\end{aligned} \tag{40}$$

For the term with phoretic mobility difference we have

$$\begin{aligned}&\frac{-1}{4\pi a^2\mathcal{D}}\int_S dS\left[\mu^{(p)}-\mu^{(a)}\right]\left[\mathcal{G}(\theta;\theta_1)-\mathcal{G}(\theta;\theta_2)\right]\hat{\boldsymbol{n}}\cdot\boldsymbol{\Gamma}^{S^+}\\ &\quad= \frac{-2\pi a^2\Gamma_0}{4\pi a^2\mathcal{D}}\left[\mu^{(p)}-\mu^{(a)}\right]\int_0^{\theta_1}\left[\mathcal{G}(\theta;\theta_1)-\mathcal{G}(\theta;\theta_2)\right]\sin\theta\,d\theta\\ &\quad- \frac{1-\cos\theta_1}{1+\cos\theta_2}\frac{-2\pi a^2\Gamma_0}{4\pi a^2\mathcal{D}}\left[\mu^{(p)}-\mu^{(a)}\right]\int_{\theta_2}^{\pi}\left[\mathcal{G}(\theta;\theta_1)-\mathcal{G}(\theta;\theta_2)\right]\sin\theta\,d\theta\end{aligned} \tag{41}$$

Using the expression (39) for the reference speed we obtain

$$\begin{aligned}
&\frac{-1}{4\pi a^2\mathcal{D}}\int_S dS\,[\mu^{(p)}-\mu^{(a)}][\mathcal{G}(\theta;\theta_1)-\mathcal{G}(\theta;\theta_2)]\,\hat{\boldsymbol{n}}\cdot\boldsymbol{\Gamma}^{S^+}\\
&=\left(\frac{\mu^{(p)}}{\mu^{(a)}}-1\right)U_{(\mathrm{ref})}\int_0^{\theta_1}[\mathcal{G}(\theta;\theta_1)-\mathcal{G}(\theta;\theta_2)]\sin\theta\,d\theta\\
&-\frac{1-\cos\theta_1}{1+\cos\theta_2}\left(\frac{\mu^{(p)}}{\mu^{(a)}}-1\right)U_{(\mathrm{ref})}\int_{\theta_2}^{\pi}[\mathcal{G}(\theta;\theta_1)-\mathcal{G}(\theta;\theta_2)]\sin\theta\,d\theta\\
&=\left(\frac{\mu^{(p)}}{\mu^{(a)}}-1\right)U_{(\mathrm{ref})}\left\{\int_0^{\theta_1}[\mathcal{G}(\theta;\theta_1)-\mathcal{G}(\theta;\theta_2)]\sin\theta\,d\theta-\frac{1-\cos\theta_1}{1+\cos\theta_2}\int_{\theta_2}^{\pi}[\mathcal{G}(\theta;\theta_1)-\mathcal{G}(\theta;\theta_2)]\sin\theta\,d\theta\right\}
\end{aligned} \tag{42}$$

Therefore, defining the dimensionless velocities,

$$U_{(\mathrm{unif})}=(2+\cos\theta_1-\cos\theta_2)\sin^2\frac{\theta_1}{2} \tag{43a}$$

$$\overset{\triangle}{U}=\int_0^{\theta_1}d\theta\sin\theta[\mathcal{G}(\theta;\theta_1)-\mathcal{G}(\theta;\theta_2)]-\frac{1-\cos\theta_1}{1+\cos\theta_2}\int_{\theta_2}^{\pi}d\theta\sin\theta[\mathcal{G}(\theta;\theta_1)-\mathcal{G}(\theta;\theta_2)], \tag{43b}$$

which are Eq. (41a) and Eq. (41b) in the main manuscript, respectively, the translational velocity along the symmetry axis becomes

$$\frac{\hat{\boldsymbol{e}}_z\cdot\boldsymbol{U}}{U_{(\mathrm{ref})}}=U_{(\mathrm{unif})}+\left(\frac{\mu^{(p)}}{\mu^{(a)}}-1\right)\overset{\triangle}{U}, \tag{44}$$

which is Eq. (40) in the main manuscript.

5. NON-AXISYMMETRIC SELF-PHORETIC PARTICLE

In this scenario, the position-dependent phoretic mobility $\mu(\theta)$ is given by

$$\mu(\theta)=\mu^{(a)}+\left[\mu^{(p)}-\mu^{(a)}\right]H(\theta-\theta_0), \tag{45}$$

and the normal component of the flux $\hat{\boldsymbol{n}}\cdot\boldsymbol{\Gamma}^{S^+}$ over the surface of the sphere is given by

$$\hat{\boldsymbol{n}}\cdot\boldsymbol{\Gamma}^{S^+}(\theta,\phi)=[1-H(\theta-\theta_0)][1-2H(\phi-\pi)]\,\Gamma_0. \tag{46}$$

We aim to obtain the translational $\boldsymbol{U}$ and rotational $\boldsymbol{\Omega}$ velocities of the self-phoretic particle.

A. Translational Velocity

The particle's translational velocity is along the y-axis, with the field kernel

$$\hat{\boldsymbol{e}}_y\cdot\mathbb{K}_t=-a\,\hat{\boldsymbol{e}}_y\cdot\nabla_S\mu+2\hat{\boldsymbol{e}}_y\cdot\hat{\boldsymbol{n}}\mu=-\hat{\boldsymbol{e}}_y\cdot\hat{\boldsymbol{e}}_\theta\,\partial_\theta\mu+2\hat{\boldsymbol{e}}_y\cdot\hat{\boldsymbol{e}}_r\mu=\sin\phi(-\cos\theta\,\partial_\theta\mu+2\mu\sin\theta) \tag{47}$$

where we have used $\hat{\boldsymbol{e}}_y \cdot \hat{\boldsymbol{e}}_\theta = \cos\theta\sin\phi$ and $\hat{\boldsymbol{e}}_y \cdot \hat{\boldsymbol{e}}_r = \sin\theta\sin\phi$. Using the expression for the phoretic mobility (45) gives

$$\begin{aligned}\hat{\boldsymbol{e}}_y \cdot \mathbb{K}_t &= \sin\phi\left\{-\cos\theta\left[\mu^{(p)} - \mu^{(a)}\right]\delta(\theta-\theta_0) + 2\left(\mu^{(a)} + \left[\mu^{(p)} - \mu^{(a)}\right] H(\theta-\theta_0)\right)\sin\theta\right\} \\ &= \sin\phi\left\{2\mu^{(a)}\sin\theta + \left[\mu^{(p)} - \mu^{(a)}\right]\left[2H(\theta-\theta_0)\sin\theta - \delta(\theta-\theta_0)\cos\theta\right]\right\} \end{aligned} \tag{48}$$

and thus, we can write the field kernel for translation along the y-axis as

$$\hat{\boldsymbol{e}}_y \cdot \mathbb{K}_t = \sin\phi\left\{\mu^{(a)}(2\sin\theta) + \left[\mu^{(p)} - \mu^{(a)}\right]\Theta_t^\Psi\right\} = f^\Psi(\theta)\sin\phi \tag{49}$$

$$\Theta_t^\Psi(\theta;\theta_0) = 2H(\theta-\theta_0)\sin\theta - \delta(\theta-\theta_0)\cos\theta \tag{50}$$

which are Eq. (47a) and Eq. (47b) in the main manuscript, respectively.

To obtain the translational flux kernel we need to calculate $\{\!\!\{\hat{\boldsymbol{e}}_y \cdot \mathbb{K}_t\}\!\!\}_{\ell m}$. The dependence of the field kernel (49) on $\sin\phi$ necessitates that the flux kernel coefficients $\{\!\!\{\hat{\boldsymbol{e}}_y \cdot \mathbb{K}_t\}\!\!\}_{\ell m}$ can be non-zero only when $m = \pm 1$, and thus $\ell \neq 0$. Using

$$\{\!\!\{\hat{\boldsymbol{e}}_y \cdot \mathbb{K}_t\}\!\!\}^*_{\ell,1} = \int_0^{2\pi}\!\!\int_0^{\pi} \hat{\boldsymbol{e}}_y \cdot \mathbb{K}_t\, Y_{\ell,1}\sin\theta\, d\theta\, d\phi = \int_0^{2\pi}\!\!\int_0^{\pi} \hat{\boldsymbol{e}}_y \cdot \mathbb{K}_t(-Y^*_{\ell,-1})\sin\theta\, d\theta\, d\phi = -\{\!\!\{\hat{\boldsymbol{e}}_y \cdot \mathbb{K}_t\}\!\!\}_{\ell,-1} \tag{51}$$

and $Y_{\ell,-1}(\theta,\phi) = -Y^*_{\ell,1}(\theta,\phi) = -Y_{\ell,1}(\theta,0)e^{-i\phi}$ we can write

$$\begin{aligned}\hat{\boldsymbol{e}}_y \cdot \boldsymbol{\mathcal{K}}_t^{S^+} &= \sum_{\ell=0}^{\infty}\sum_{-\ell\le m\le \ell}\frac{1}{\ell+1}\{\!\!\{\hat{\boldsymbol{e}}_y \cdot \mathbb{K}_t\}\!\!\}_{\ell m} Y_{\ell,m}(\theta,\phi) \\ &= \sum_{\ell=1}^{\infty}\frac{1}{\ell+1}\left[\{\!\!\{\hat{\boldsymbol{e}}_y \cdot \mathbb{K}_t\}\!\!\}_{\ell,-1} Y_{\ell,-1}(\theta,\phi) + \{\!\!\{\hat{\boldsymbol{e}}_y \cdot \mathbb{K}_t\}\!\!\}_{\ell,1} Y_{\ell,1}(\theta,\phi)\right] \\ &= \sum_{\ell=1}^{\infty}\frac{1}{\ell+1}\left[\{\!\!\{\hat{\boldsymbol{e}}_y \cdot \mathbb{K}_t\}\!\!\}_{\ell,1} e^{i\phi} - \{\!\!\{\hat{\boldsymbol{e}}_y \cdot \mathbb{K}_t\}\!\!\}_{\ell,-1} e^{-i\phi}\right] Y_{\ell,1}(\theta,0) \\ &= \sum_{\ell=1}^{\infty}\frac{1}{\ell+1}\left[\{\!\!\{\hat{\boldsymbol{e}}_y \cdot \mathbb{K}_t\}\!\!\}_{\ell,1} e^{i\phi} + \{\!\!\{\hat{\boldsymbol{e}}_y \cdot \mathbb{K}_t\}\!\!\}^*_{\ell,1} e^{-i\phi}\right] Y_{\ell,1}(\theta,0) \\ &= \sum_{\ell=1}^{\infty}\frac{1}{\ell+1} 2\mathrm{Re}\left[\{\!\!\{\hat{\boldsymbol{e}}_y \cdot \mathbb{K}_t\}\!\!\}_{\ell,1} e^{i\phi}\right] Y_{\ell,1}(\theta,0) \end{aligned} \tag{52}$$

where Re[·] returns the real part of the complex number. Evaluating

$$\begin{aligned} e^{i\phi}\{\!\!\{\hat{\boldsymbol{e}}_y \cdot \mathbb{K}_t\}\!\!\}_{\ell,1} &= e^{i\phi}\{\!\!\{f^\Psi(\theta)\sin\phi\}\!\!\}_{\ell,1} \\ &= e^{i\phi}\int_0^{2\pi}\int_0^{\pi} f^\Psi(\theta')\sin\phi'\, Y^*_{\ell,1}(\theta',\phi')\sin\theta' d\theta' d\phi' \\ &= e^{i\phi}\int_0^{2\pi} e^{-i\phi'}\sin\phi' d\phi' \int_0^{\pi} f^\Psi(\theta')\, Y_{\ell,1}(\theta',0)\sin\theta' d\theta' \\ &= (\cos\phi + i\sin\phi)(-i\pi)\int_0^{\pi} f^\Psi(\theta')\, Y_{\ell,1}(\theta',0)\sin\theta' d\theta' \\ &= \pi(-i\cos\phi + \sin\phi)\int_0^{\pi} f^\Psi(\theta')\, Y_{\ell,1}(\theta',0)\sin\theta' d\theta' \end{aligned} \tag{53}$$

and taking its reals part, combined with Eq. (52), leads to

$$\hat{\boldsymbol{e}}_y \cdot \boldsymbol{\mathcal{K}}_t^{S^+} = \sin\phi \sum_{\ell=1}^{\infty} \frac{2\pi}{\ell+1} Y_{\ell,1}(\theta,0) \int_0^{\pi} f^{\Psi}(\theta')\, Y_{\ell,1}(\theta',0) \sin\theta' d\theta' \tag{54}$$

and we are left with the calculation of the integral. For the uniform phoretic mobility term $\mu^{(a)}(2\sin\theta)$, using $\sin\theta = BY_{1,1}(\theta,0)$, where $B = -2\sqrt{2\pi/3}$, we have

$$\begin{aligned} Y_{\ell,1}(\theta,0) \int_0^{\pi} [\mu^{(a)}(2\sin\theta')]\, Y_{\ell,1}(\theta',0) \sin\theta' d\theta' &= 2\mu^{(a)} Y_{\ell,1}(\theta,0) \int_0^{\pi} BY_{1,1}(\theta',0)\, Y_{\ell,1}(\theta',0) \sin\theta' d\theta' \\ &= 2\mu^{(a)} Y_{\ell,1}(\theta,0) \left(\frac{B}{2\pi}\delta_{\ell,1}\right) \\ &= 2\frac{1}{2\pi}\mu^{(a)} BY_{1,1}(\theta,0)\delta_{\ell,1} \\ &= \frac{1}{\pi}\mu^{(a)} \sin\theta\, \delta_{\ell,1} \end{aligned} \tag{55}$$

and thus,

$$\sum_{\ell=1}^{\infty} \frac{2\pi}{\ell+1} Y_{\ell,1}(\theta,0) \int_0^{\pi} [\mu^{(a)}(2\sin\theta')]\, Y_{\ell,1}(\theta',0) \sin\theta' d\theta' = \sum_{\ell=1}^{\infty} \frac{2\pi}{\ell+1} \frac{1}{\pi} \mu^{(a)} \sin\theta\, \delta_{\ell,1} = \mu^{(a)} \sin\theta \tag{56}$$

For the term corresponding to the difference of the phoretic mobilities we need to use direct integration. Therefore, defining

$$\Theta_t^{\boldsymbol{\Gamma}}(\theta;\theta_0) = \sum_{l=1}^{\infty} \frac{2\pi}{\ell+1} \left\{ 2\int_{\theta_0}^{\pi} Y_{\ell,1}(\theta',0)(\sin\theta')^2\, d\theta' - \sin\theta_0 \cos\theta_0 Y_{\ell,1}(\theta_0,0) \right\} Y_{l,1}(\theta,0) \tag{57}$$

which is Eq. (48b) in the main manuscript, we can write the y-component of the flux kernel in the form,

$$\hat{\boldsymbol{e}}_y \cdot \boldsymbol{\mathcal{K}}_t^{S^+} = \sin\phi \left\{ \mu^{(a)} \sin\theta + \left[\mu^{(p)} - \mu^{(a)}\right] \Theta_t^{\boldsymbol{\Gamma}}(\theta;\theta_0) \right\} = f^{\Gamma}(\theta)\, \sin\phi \tag{58}$$

which is Eq. (48a) in the main manuscript.

Therefore, we can obtain the translational velocity,

$$\begin{aligned}
\hat{\boldsymbol{e}}_y \cdot \boldsymbol{U} &= \frac{-1}{4\pi a^2 \mathcal{D}} \int_S dS\, \hat{\boldsymbol{e}}_y \cdot \boldsymbol{\mathcal{K}}_t^{S^+} \hat{\boldsymbol{n}} \cdot \boldsymbol{\Gamma}^{S^+} \\
&= \frac{-1}{4\pi a^2 \mathcal{D}}\, a^2 \int_0^{2\pi} \int_0^{\pi} \sin\theta\, d\theta\, d\phi \left[f^{\Gamma}(\theta)\, \sin\phi\right] \{[1 - H(\theta-\theta_0)][1 - 2H(\phi - \pi)]\,\Gamma_0\} \\
&= \frac{-\Gamma_0}{4\pi \mathcal{D}} \int_0^{2\pi} [1 - 2H(\phi-\pi)] \sin\phi d\phi \int_0^{\pi} \sin\theta\, d\theta\, f^{\Gamma}(\theta)[1 - H(\theta - \theta_0)] \\
&= \frac{-\Gamma_0}{4\pi \mathcal{D}} \times 4 \times \int_0^{\theta_0} \sin\theta\, d\theta\, f^{\Gamma}(\theta) \\
&= \frac{U_{(\mathrm{ref})}}{\mu^{(a)}} \frac{2}{\pi} \int_0^{\theta_0} \sin\theta\, d\theta\, f^{\Gamma}(\theta) \\
&= \frac{U_{(\mathrm{ref})}}{\mu^{(a)}} \frac{2}{\pi} \int_0^{\theta_0} \sin\theta\, d\theta\, \left\{\mu^{(a)} \sin\theta + \left[\mu^{(p)} - \mu^{(a)}\right] \Theta_t^{\boldsymbol{\Gamma}}(\theta;\theta_0)\right\} \\
&= U_{(\mathrm{ref})} \frac{2}{\pi} \int_0^{\theta_0} \sin^2\theta\, d\theta + U_{(\mathrm{ref})} \left[\frac{\mu^{(p)}}{\mu^{(a)}} - 1\right] \frac{2}{\pi} \int_0^{\theta_0} \sin\theta\, d\theta\, \Theta_t^{\boldsymbol{\Gamma}}(\theta;\theta_0) \\
&= U_{(\mathrm{ref})} \frac{2}{\pi} \frac{1}{2} \left(\theta_0 - \frac{1}{2}\sin 2\theta\right) + U_{(\mathrm{ref})} \left[\frac{\mu^{(p)}}{\mu^{(a)}} - 1\right] \frac{2}{\pi} \int_0^{\theta_0} \sin\theta\, d\theta\, \Theta_t^{\boldsymbol{\Gamma}}(\theta;\theta_0) \\
&= U_{(\mathrm{ref})} \left(\frac{\theta_0}{\pi} - \frac{\sin 2\theta}{2\pi}\right) + U_{(\mathrm{ref})} \left[\frac{\mu^{(p)}}{\mu^{(a)}} - 1\right] \frac{2}{\pi} \int_0^{\theta_0} \sin\theta\, d\theta\, \Theta_t^{\boldsymbol{\Gamma}}(\theta;\theta_0) \qquad (59)
\end{aligned}$$

Therefore, defining the dimensionless velocities,

$$U_{(\mathrm{unif})} = \frac{\theta_0}{\pi} - \frac{\sin 2\theta_0}{2\pi}, \qquad (60)$$

$$\overset{\triangle}{U} = \frac{2}{\pi} \int_0^{\theta_0} d\theta\, \sin\theta\, \Theta_t^{\boldsymbol{\Gamma}}(\theta;\theta_0), \qquad (61)$$

which are Eq. (50a) and Eq. (50b) in the main manuscript, respectively, we have the scaled translational velocity along the y direction,

$$\frac{\hat{\boldsymbol{e}}_y \cdot \boldsymbol{U}}{U_{(\mathrm{ref})}} = U_{(\mathrm{unif})} + \left(\frac{\mu^{(p)}}{\mu^{(a)}} - 1\right) \overset{\triangle}{U}, \qquad (62)$$

which is Eq. (49) in the main manuscript.

B. Rotational Velocity

For the rotational field kernel, we have

$$\mathbb{K}_r = -\hat{\boldsymbol{n}} \times (a \nabla_S \mu) = -\hat{\boldsymbol{e}}_r \times \hat{\boldsymbol{e}}_\theta \partial_\theta \mu = -\hat{\boldsymbol{e}}_\phi \partial_\theta \mu = -\hat{\boldsymbol{e}}_\phi \left[\mu^{(p)} - \mu^{(a)}\right] \delta(\theta - \theta_0) \qquad (63)$$

which is Eq. (51a) in the main manuscript. Since the only ϕ dependence is in $\hat{\boldsymbol{e}}_\phi$, only $\{\!\!\{\mathbb{K}_r\}\!\!\}_{\ell,\pm1}$ terms can be non-zero, and we have

$$\boldsymbol{\mathcal{K}}_r^{S^+} = \sum_{\ell=0}^{\infty} \sum_{-\ell\leq m\leq\ell} \frac{1}{\ell+1} \{\!\!\{\mathbb{K}_r\}\!\!\}_{\ell m}\, Y_{\ell,m}(\theta,\phi) = \sum_{\ell=1}^{\infty} \frac{1}{\ell+1} 2\mathrm{Re}\left[\{\!\!\{\mathbb{K}_r\}\!\!\}_{\ell,1}\, e^{i\phi}\right] Y_{\ell,1}(\theta,0) \tag{64}$$

Evaluating

$$\begin{aligned}
\{\!\!\{\hat{\boldsymbol{e}}_\phi\, \delta(\theta-\theta_0)\}\!\!\}_{\ell,1} &= \int_0^{\pi}\int_0^{2\pi} \hat{\boldsymbol{e}}_\phi \delta(\theta-\theta_0)\, Y_{\ell,1}^*(\theta,\phi)\sin\theta\, d\theta\, d\phi \\
&= \int_0^{2\pi} e^{-i\phi}\hat{\boldsymbol{e}}_\phi\, d\phi \int_0^{\pi} \delta(\theta-\theta_0)\, Y_{\ell,1}(\theta,0)\sin\theta\, d\theta \\
&= \int_0^{2\pi} e^{-i\phi}(-\sin\phi\,\hat{\boldsymbol{e}}_x + \cos\phi\,\hat{\boldsymbol{e}}_y)\, d\phi \int_0^{\pi} \delta(\theta-\theta_0)\, Y_{\ell,1}(\theta,0)\sin\theta\, d\theta \\
&= \pi(i\hat{\boldsymbol{e}}_x + \hat{\boldsymbol{e}}_y) Y_{\ell,1}(\theta_0,0)\sin\theta_0
\end{aligned} \tag{65}$$

and taking its real part leads to

$$\begin{aligned}
2\mathrm{Re}\left[\{\!\!\{\mathbb{K}_r\}\!\!\}_{\ell,1}\, e^{i\phi}\right] &= -2\pi\left[\mu^{(p)} - \mu^{(a)}\right] Y_{\ell,1}(\theta_0,0)\sin\theta_0\, \mathrm{Re}\left[(i\hat{\boldsymbol{e}}_x + \hat{\boldsymbol{e}}_y)e^{i\phi}\right] \\
&= -2\pi\left[\mu^{(p)} - \mu^{(a)}\right] Y_{\ell,1}(\theta_0,0)\sin\theta_0\,(-\sin\phi\,\hat{\boldsymbol{e}}_x + \cos\phi\,\hat{\boldsymbol{e}}_y) \\
&= -2\pi\left[\mu^{(p)} - \mu^{(a)}\right] Y_{\ell,1}(\theta_0,0)\sin\theta_0\,\hat{\boldsymbol{e}}_\phi
\end{aligned} \tag{66}$$

and thus,

$$\boldsymbol{\mathcal{K}}_r^{S^+} = -\hat{\boldsymbol{e}}_\phi\left[\mu^{(p)} - \mu^{(a)}\right] \sum_{\ell=1}^{\infty} \frac{2\pi}{\ell+1} Y_{\ell,1}(\theta_0,0)\, \sin\theta_0\, Y_{\ell,1}(\theta,0) \tag{67}$$

Thus, defining

$$\Theta_r^{\boldsymbol{\Gamma}}(\theta;\theta_0) = 2\pi \sum_{\ell=1}^{\infty} Y_{\ell,1}(\theta_0,0) Y_{\ell,1}(\theta,0)\left[\frac{\sin\theta_0}{\ell+1}\right] \tag{68}$$

which is Eq. (52b) in the main manuscript, we have

$$\boldsymbol{\mathcal{K}}_r^{S^+} = -\hat{\boldsymbol{e}}_\phi\left[\mu^{(p)} - \mu^{(a)}\right]\Theta_r^{\boldsymbol{\Gamma}}(\theta;\theta_0). \tag{69}$$

which is Eq. (51b) in the main manuscript.

Calculating the angular velocity is straightforward,

$$\begin{aligned}
\boldsymbol{\Omega} &= \frac{-3}{8\pi a^3 \mathcal{D}} \int_{S^+} dS\, \boldsymbol{\mathcal{K}}_r^{S^+}\, \hat{\boldsymbol{n}} \cdot \boldsymbol{\Gamma}^{S^+} \\
&= \frac{-3}{8\pi a^3 \mathcal{D}} \left[\mu^{(p)} - \mu^{(a)}\right] a^2 \int_0^{2\pi} \int_0^{\pi} \sin\theta\, d\theta\, d\phi \left[-\hat{\boldsymbol{e}}_\phi \Theta_r^{\boldsymbol{\Gamma}}(\theta;\theta_0)\right] \{[1 - H(\theta-\theta_0)][1 - 2H(\phi-\pi)]\,\Gamma_0\} \\
&= \frac{-3\Gamma_0}{8\pi a \mathcal{D}} \left[\mu^{(p)} - \mu^{(a)}\right] \int_0^{2\pi} [1 - 2H(\phi-\pi)](-\hat{\boldsymbol{e}}_\phi)\, d\phi \int_0^{\pi} \sin\theta\, d\theta\, \Theta_r^{\boldsymbol{\Gamma}}(\theta;\theta_0)[1 - H(\theta-\theta_0)] \\
&= \frac{-3\Gamma_0}{8\pi a \mathcal{D}} \left[\mu^{(p)} - \mu^{(a)}\right] \times 4\hat{\boldsymbol{e}}_x \times \int_0^{\theta_0} \sin\theta\, d\theta\, \Theta_r^{\boldsymbol{\Gamma}}(\theta;\theta_0) \\
&= \hat{\boldsymbol{e}}_x \frac{U_{(\mathrm{ref})}}{a} \left[\frac{\mu^{(p)}}{\mu^{(a)}} - 1\right] \frac{3}{\pi} \int_0^{\theta_0} \sin\theta\, d\theta\, \Theta_r^{\boldsymbol{\Gamma}}(\theta;\theta_0)
\end{aligned} \tag{70}$$

Therefore, defining the reference angular velocity scale as

$$\Omega_{(\mathrm{ref})} = U_{(\mathrm{ref})}/a, \tag{71}$$

the scaled angular velocity as

$$\frac{\boldsymbol{\Omega}}{\Omega_{(\mathrm{ref})}} = \hat{\boldsymbol{e}}_x \left(\frac{\mu^{(p)}}{\mu^{(a)}} - 1\right) \overset{\triangle}{\Omega}(\theta_0), \tag{72}$$

which is Eq. (54) in the main manuscript. The dimensionless function is

$$\overset{\triangle}{\Omega}(\theta_0) = \frac{3}{\pi} \int_0^{\theta_0} d\theta\, \sin\theta\, \Theta_r^{\boldsymbol{\Gamma}}(\theta;\theta_0), \tag{73}$$

which is Eq. (55) in the main manuscript.