Probing the physical origins of droplet friction using a critically damped cantilever

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S1. LIST OF SUPPORTING VIDEOS

Video 1. A 10 µl water droplet moving on a superhydrophobic surface at a speed $U = 0.1$ mm s^{-1} is always in contact with the topmost solid fractions, i.e., Cassie-Baxter state. At the receding end, we observe jagged, discontinuous contact line, indicating the presence of pinning. In contrast, at the advancing front, contact line is smooth and circular, indicating an advancing contact angle close to 180°.

Video 2. Transition from stationary to a speed of $U = 30$ cm s^{-1} for the same 10 µl droplet. Initially the droplet is in contact with the topmost solid fractions. At a later part, the droplet starts to lift off the surface.

Video 3. Steady motion at $U = 30$ cm s^{-1} for the same 10 µl droplet. The droplet completely lifts off the surface. Contact line is smooth and circular at both the advancing front and receding end. Contact-line pinning is unimportant.

Video 4. Once the motion is stopped for the same 10 µl droplet, air film beneath the droplet drains away, and the droplet is again in contact with the topmost solid fractions.

S2. NATURAL FREQUENCY AND SPRING CONSTANT OF CANTILEVER

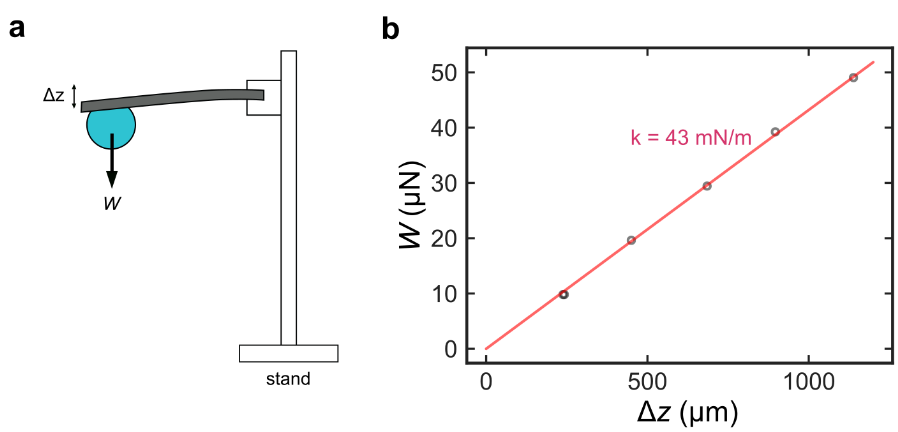

FIG. S1. *Added weight method.* (a) Spring constant k can be determined by measuring the cantilever deflection Δz after adding water droplets of different weight W. (b) Plot of W against Δz. Slope corresponds to k.

The flexular spring constant of the cantilever k can be determined using the added weight method (**Fig. S1a**) [1, 2]. We placed masses (e.g., microlitre-sized water droplets) of different weights W at the cantilever tip and measured the deflection Δz optically. The slope of W against Δz corresponds to k, since $W = k\Delta z$. We obtained $k = 43 \pm 2$ mN m^{-1} for a cantilever with length $L = 50 \pm 1$ mm (**Fig. S1b**).

Alternatively, we can deduce the spring constant from the cantilever natural frequency with no mass added. A cantilever with a uniform mass per unit length μ has a natural frequency ω_o given by

$$\omega_o = 2\pi f_o = \frac{1.875^2}{L^2}\sqrt{\frac{EI}{\mu}}, \tag{S1}$$

where L is the cantilever length, E is the Young's modulus, and I is the second moment of inertia. At the same time,

$$k = \sqrt{\frac{3EI}{L^3}}. \tag{S2}$$

By comparing Equations S1 and S2, we can relate ω_o and k

$$\omega_o = 2.03\sqrt{\frac{k}{m_c}}, \tag{S3}$$

where $m_c = \mu L$ is the mass of the cantilever. Note that Equation S3 differs slightly from the more familiar $\omega_o = \sqrt{k/m}$ for a simple harmonic oscillator, because the mass m_c is distributed uniformly along the cantilever rather than concentrated at one end.

The simplest way to obtain ω_o (and hence k) is to apply an impulse to one end of the cantilever, for example, by using a piezo-actuator (**Fig. S2a**) or by using a ball blower. One end of the cantilever is attached to the piezo-actuator, and the cantilever displacement at the attached end $\Delta x'$ can be controlled by applying different voltages V (step function) on the piezo-actuator. This causes the free-end of the cantilever to oscillate with a frequency f' close to the natural frequency f_o and a magnitude Δx that decays with a characteristic decay rate β, i.e.,

$$\Delta x = \Delta x_o \exp(-\beta t)\cos(\omega' t - \phi), \tag{S4}$$

where Δx_o is the maximum deflection amplitude, ϕ is phase of the oscillation, and $\omega' = 2\pi f' = \sqrt{\omega_o^2 - \beta^2}$ is the observed frequency. Here, we have assumed that β is small and the system is underdamped, and if $\beta \ll \omega_o$ we can make the approximation $\omega' = \omega_o$. **Figure S2b, c** shows the free decay for the cantilever of length $L = 50$ mm (same as in **Figure S1**) and mass $m_c = 2.7 \pm 0.2$ mg for two different $\Delta x_o = 12$ and 23 µm, respectively. We found that $f_o = 42.6 \pm 0.1$

Hz and $\beta = 13.0 \pm 0.3$ s^{-1}, which translates to the spring constant $k = 0.243\ m\omega_o^2 = 47 \pm 1$ mN m^{-1}.

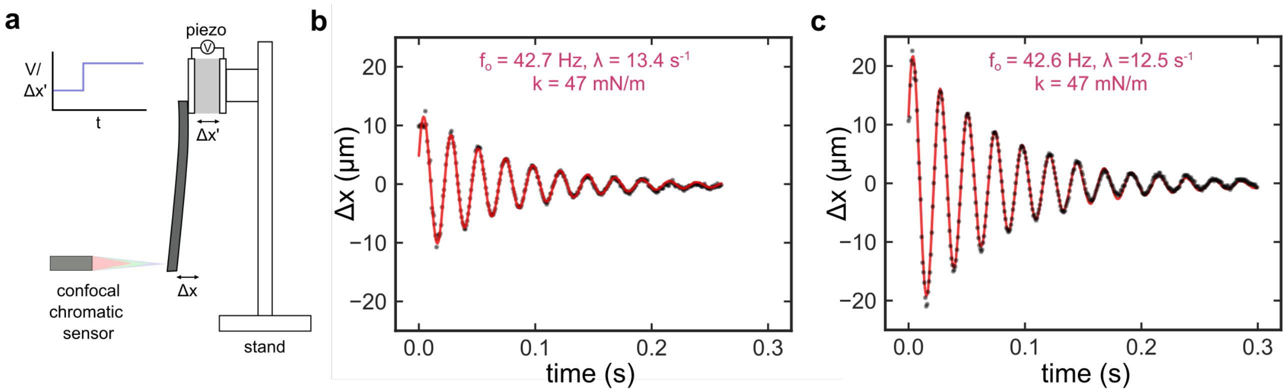

FIG. S2. *Free-decay method.* (a) To determine the natural frequency f_o of the cantilever, we use a piezo-actuator to give an impulse (of displacement $\Delta x'$) at one end of the cantilever. The deflection at the free end Δx can be monitored using a confocal chromatic sensor. The free end will oscillate at a frequency close to f_o and with a characteristic decay rate β, as shown in (b) and (c) for two different $\Delta x'$. Dots are experimental data, while red line is the best-fit for Equation S4.

The values of k obtained by the two methods outlined above are consistent with one another, as summarized in Table S1. The spring constant is not affected by the addition of the damping system described in this manuscript.

TABLE S1. Determining cantilever spring constant k using different methods

Methods	f_o (Hz)	β (s^{-1})	k (mN m^{-1})
Added weight	-	-	43 $\pm$ 2
Free decay	42.6 $\pm$ 0.1	13.0 $\pm$ 0.3	47 $\pm$ 1

S3. EQUATION OF MOTION FOR DIFFERENT DAMPINGS

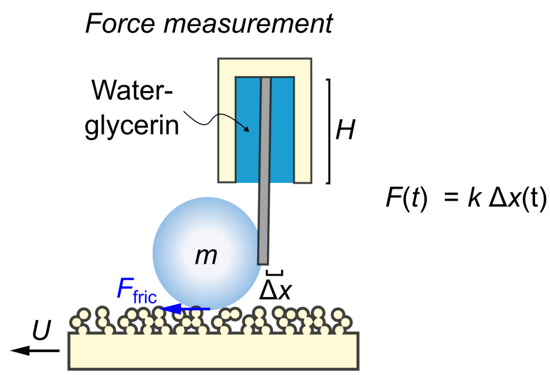

FIG. S3. Schematic of a water droplet during each cycle of force measurement (in one direction).

The equation of motion for the droplet of mass m is given by

$$m\,\Delta x(t)'' + b\,\Delta x(t)' + k\,\Delta x(t) = F_{\mathrm{fric}} \tag{S5}$$

where $\Delta x(t)$ is the cantilever displacement and $k\,\Delta x(t)$ corresponds to the force measured by the cantilever system $F(t)$, i.e., the force response function in the main text (Fig. 3). In the limit of long time $t \gg \tau$ where τ is some characteristic response time, $\Delta x(t) \to F_{\mathrm{fric}}/k$ and $F(t) \to F_{\mathrm{fric}}$. However, the evolution of $\Delta x(t)$ and hence $F(t)$ with time t depends on the amount of damping, in particular on $b/\sqrt{4km}$.

For an underdamped system $b/\sqrt{4km} < 1$, the solution is given by

$$\Delta x(t) = \Delta x_0\, e^{-(b/2m)\,t} \cos(\omega' t + \phi) + F_{\mathrm{fric}}/k \tag{S6}$$

where $\omega' = \sqrt{\omega_o^2 - b^2/4m^2} \approx \omega_o$ is the oscillation frequency and where $\omega_o = \sqrt{k/m}$ is the cantilever's natural frequency. Δx_0 and ϕ are fitting parameters that depend on initial conditions. Equivalently,

$$F(t) = F_0 e^{-(b/2m)\,t} \cos(\omega_o t + \phi) + F_{\mathrm{fric}} \tag{S7}$$

The force curve will therefore dampens with an exponential decay envelope $\exp(-t/\tau)$, where

$\tau = 2m/b$ is the characteristic decay/response time.

For a critically damped system $b/\sqrt{4km} = 1$, the cantilever system will respond with the fastest theoretical response time $\tau = 1/\omega_o$ such that

$$\begin{aligned} F(t) &= (F_0 + F_1 t)\, e^{-\omega_o t} + F_{\text{fric}} \\ &\approx F_0 \exp(-\omega_o t) + F_{\text{fric}} \quad \text{for } t \gg 1/\omega_o \end{aligned} \tag{S8}$$

where F_0 and F_1 are again fitting parameters.

Increasing the damping further to $b/\sqrt{4km} > 1$ results in an overdamped system such that

$$F(t) = F_0 \exp(-r_0 t) + F_1 \exp(-r_1 t) + F_{\text{fric}} \tag{S9}$$

where

$$\begin{aligned} r_0 &= \frac{b}{2m}\left(1 - \sqrt{1 - \frac{4km}{b^2}}\right) \\ &\approx \frac{k}{b} \qquad \text{for } b/\sqrt{4km} \gg 1 \end{aligned} \tag{S10}$$

and

$$\begin{aligned} r_1 &= \frac{b}{2m}\left(1 + \sqrt{1 - \frac{4km}{b^2}}\right) \\ &\approx \frac{b}{m} \qquad \text{for } b/\sqrt{4km} \gg 1 \end{aligned} \tag{S11}$$

. Since $r_1 \gg r_0$, Equation S9 can be simplified to

$$\begin{aligned} F(t) &\approx F_0\, e^{-r_0 t} + F_{\text{fric}} \quad \text{for } t \gg m/b \\ &\approx F_0\, e^{-(k/b)t} + F_{\text{fric}} \end{aligned} \tag{S12}$$

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The force response function can therefore be well-approximated as

$$F(t) \approx \begin{cases} F_0\, e^{-(b/2m)t}\cos(\omega_o t + \phi) + F_{\text{fric}} & \text{for } \frac{b}{\sqrt{4mk}} \ll 1 \\ F_0\, e^{-\omega_o t} + F_{\text{fric}} & \text{for } \frac{b}{\sqrt{4mk}} \approx 1 \\ F_0\, e^{-(k/b)t} + F_{\text{fric}} & \text{for } \frac{b}{\sqrt{4mk}} \gg 1 \end{cases} \tag{S13}$$

with a response time of

$$\tau \approx \begin{cases} 2m/b & \text{for } \frac{b}{\sqrt{4mk}} \ll 1 \\ 1/\omega_o & \text{for } \frac{b}{\sqrt{4mk}} \approx 1 \\ b/k & \text{for } \frac{b}{\sqrt{4mk}} \gg 1 \end{cases} \tag{S14}$$

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S4. RICM IMAGES

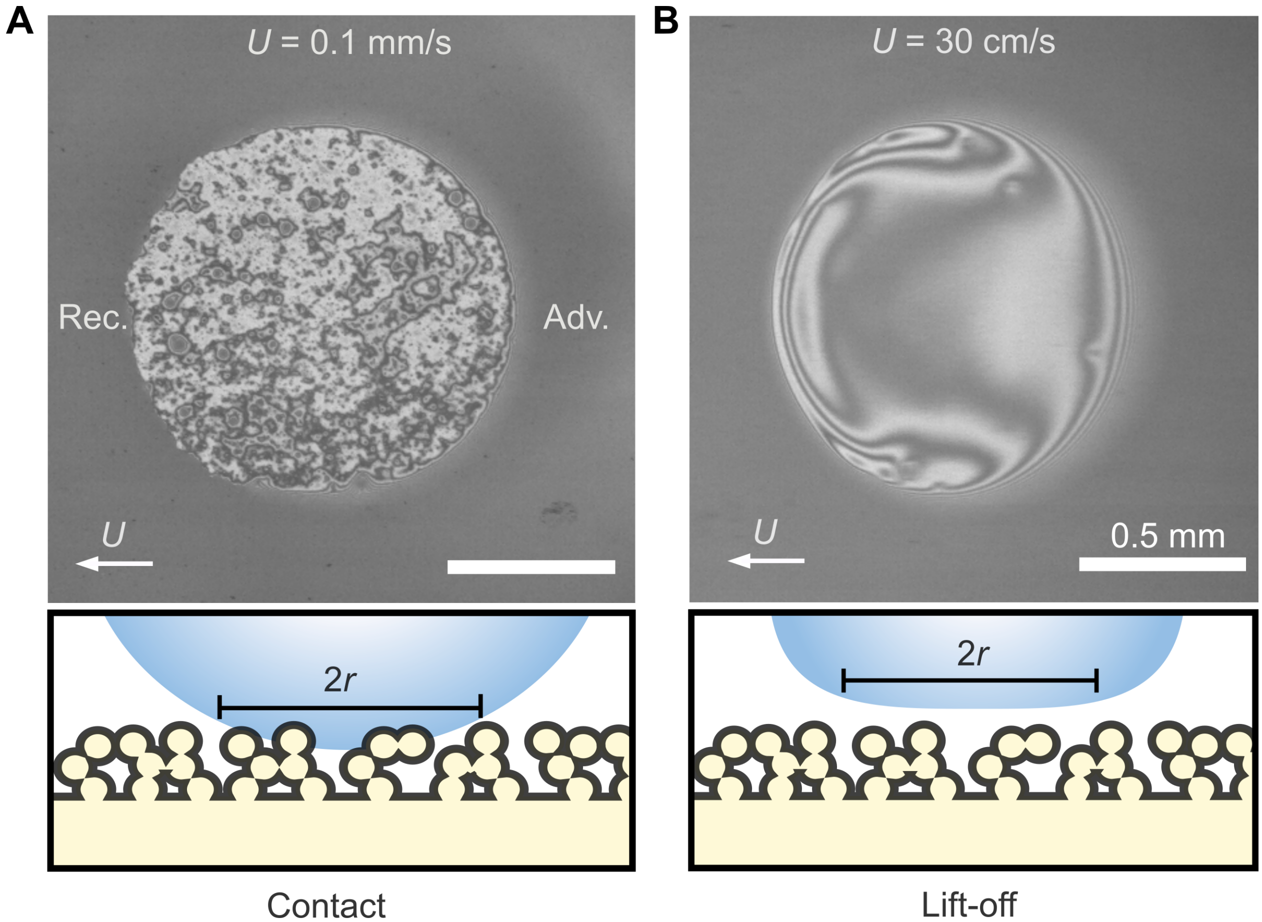

FIG. S4. *Droplet base visualized using Reflection Interference Contrast Microscopy.* (A) At low U = 0.1 mm s^{-1}, droplet is in contact with the topmost solid fractions. The advancing (Adv.) front and the receding (Rec.) ends are indicated on the image. Contact-line is not smooth at the receding end. (B) In contrast, at high U = 30 cm s^{-1}, droplet lifts off from the surface. Droplet base profile is smooth and circular. Arrows indicate motion of the surface. Scale bars for **A** and **B** are 0.5 mm.

S5. COMPARISON WITH LITERATURE DATA

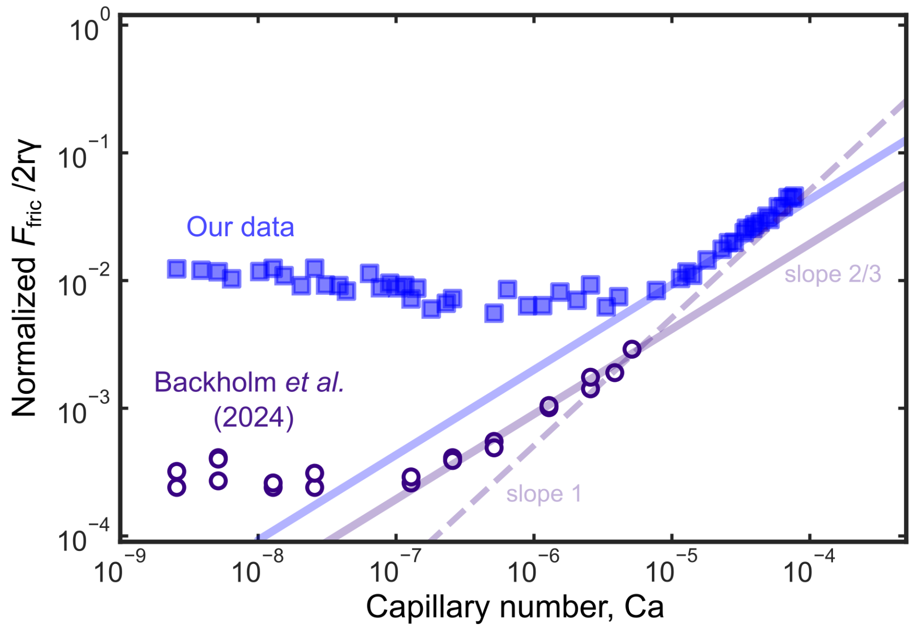

FIG. S5. *Comparison with literature data.* Data from Fig. 5a in Backholm *et al.* (2024) (unfilled circles) are superimposed with our data (filled square markers). A scaling of $F_{\mathrm{crit}} \propto U^{2/3}$ in the limit of high speed U explains both datasets very well.

Backholm *et al.* (2024) recently reported a similar increase in F_{crit} with U on superhydrophobic surface when $U > 1$ cm s^{-1}, and they propose a scaling of $F_{\mathrm{crit}} \propto U$. We have superimposed their data (in non-dimensional forms $F_{\mathrm{crit}}/2r\gamma$ vs. Ca) with ours in Fig. S5. A scaling of $F_{\mathrm{crit}} \propto U^{2/3}$ clearly explains both datasets much better than $F_{\mathrm{crit}} \propto U$.

[1] M. Backholm and O. Bäumchen, "Micropipette force sensors for in vivo force measurements on single cells and multicellular microorganisms," Nat. Protoc. **14**, 594–615 (2019).

[2] D. Daniel, J. V. I. Timonen, R. Li, S. J. Velling, and J. Aizenberg, "Oleoplaning droplets on lubricated surfaces," Nat. Phys. **13**, 1020–1025 (2017).

[3] M. Backholm, T. Kärki, H. A. Nurmi, M. Vuckovac, V. Turkki, S. Lepikko, V. Jokinen, D. Quéré, J. V. I. Timonen, and R. H. A. Ras, "Toward vanishing droplet friction on repellent surfaces," Proc. Natl. Acad. Sci. USA **121**, e2315214121 (2024).